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Preface

In spite of the large number of published mathematical tables, until the appearance of *A Handbook of Integer Sequences* (HIS) in 1973 there was no table of sequences of integers. Thus someone coming across the sequence 1, 1, 2, 5, 15, 52, 203, 877, 4140, . . . , for example, would have had difficulty in finding out that these are the Bell numbers, that they have been extensively studied, and that they can be generated by expanding e^{e^x-1} in powers of x. The 1973 book remedied this situation to a certain extent, and the *Encyclopedia of Integer Sequences* is a greatly expanded version of that book. The main table now contains 5488 sequences of integers (compared with 2372 in the first book), collected from all branches of mathematics and science. The sequences are arranged in numerical order, and for each one a brief description and a reference are given. Figures interspersed throughout the table illustrate the most important sequences. The first part of the book describes how to use the table and gives methods for analyzing unknown sequences.

Who will use this book? Anyone who has ever been confronted with a strange sequence, whether in an intelligence test in high school, e.g.,

$$1, 11, 21, 1211, 111221, 312211, 13112221, \ldots$$

(guess![1]), or in solving a mathematical problem, e.g.,

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$$

(the Catalan numbers), or from a counting problem, e.g.,

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, \ldots$$

(the number of rooted trees with n nodes), or in computer science, e.g.,

$$0, 1, 3, 5, 9, 11, 14, 17, 25, 27, \ldots$$

(the number of comparisons needed to sort n elements by list merging), or in physics, e.g.,

$$1, 6, 30, 138, 606, 2586, \ldots$$

[1]For many more terms and the explanation, see the main table.

(susceptibility coefficients for the planar hexagonal lattice[2]), or in chemistry, e.g.,

$$1, 1, 4, 8, 22, 51, 136, 335, 871, 2217, \ldots$$

(the number of alkyl derivatives of benzene with $n = 6, 7, \ldots$ carbon atoms), or in electrical engineering, e.g.,

$$3, 7, 46, 4436, 134281216, \ldots$$

(the number of Boolean functions of n variables), will find this encyclopedia useful.

If you encounter an integer sequence at work or at play and you want to find out if anyone has ever come across it before and, if so, how it is generated, then this is the book you need!

In addition to identifying integer sequences, the *Encyclopedia* will serve as an index to the literature for locating references on a particular problem and for quickly finding numbers like the number of partitions of 30, the 18th Catalan number, the expansion of π to 60 decimal places, or the number of possible chess games after 8 moves. It might also be useful to have around when the first signals arrive from Betelgeuse (sequence M5318, for example, would be a friendly beginning).

Some quotations from letters will show the diversity and enthusiasm of readers of the 1973 book. We expect the new book will find even wider applications, and look forward to hearing from readers who have used it successfully.

"I recently had the occasion to look for a sequence in your book. It wasn't there. I tried the sequence of first differences. It *was* there and pointed me in the direction of the literature. Enchanting" (Herbert S. Wilf, University of Pennsylvania).

"I also found N. J. A. Sloane's *A Handbook of Integer Sequences* to be an invaluable tool. I shall say no more about this marvelous reference except that every recreational mathematician should buy a copy forthwith" (Martin Gardner, *Scientific American,* July 1974).

[2]Also called the triangular lattice.

"Incomparable, eccentric, yet very useful. Contains thousands of 'well-defined and interesting' infinite sequences together with references for each. Sequences are arranged lexicographically and (to minimize errors) typeset from computer tape. If you ever wondered what comes after 1, 2, 4, 8, 17, 35, 71, ..., this is the place to look it up" (Lynn A. Steen, Telegraphic Review, *American Mathematical Monthly*, April 1974).

Nontechnical readers wrote to bless us, to speak of reading the book "cover to cover," or to remark that it was getting a great deal of use to the detriment of household chores and so on. Specialists in various fields had other tales to tell.

Anthony G. Shannon, an Australian combinatorial mathematician, wrote: "I must say how impressed I am with the book and already I am insisting that my students know their way around it just as with classics such as Abramowitz and Stegun."

Researchers wrote: "Our process of discovery consisted of generating these sequences and then identifying them with the aid of Sloane's *Handbook of Integer Sequences*" (J. M. Borwein, P. B. Borwein, and K. Dilcher, *American Mathematical Monthly*, October 1989).

Allen J. Schwenk, a graph theorist in Maryland wrote: "I thought I had something new until your book sent me to the Riordan reference, where I found 80% of my results and so I abandoned the problem."

We received letters describing the usefulness of the *Handbook* from such diverse readers as: a German geophysicist, a West Virginian astronomer, various graduate students, physicists, and even an epistemologist.

Finally, Harvey J. Hindin, writing from New York concluded a letter by saying:

> "There's the Old Testament, the New Testament, and the *Handbook of Integer Sequences*."

Abbreviations

Abbreviations for the references are listed in the bibliography. References to journals give volume, page number, year.

$a(n)$	nth term of sequence
$A(x)$	generating function for sequence, usually the ordinary generating function $A(x) = \sum a_n x^n$, occasionally the exponential generating function $A_E(x) = \sum a_n x^n/n!$
AND	logical "AND", sometimes applied to binary representations of numbers
B_n	Bernoulli number (see Fig. M4189)
b.c.c .	body-centered cubic lattice (see [SPLAG 116])
binomial transform	of sequence $a_0, a_1, \ldots$ is sequence $b_0, b_1, \ldots$ where $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$
$C(n)$ or C_n	nth Catalan number (see Fig. M1459)
$C(n,k)$ or $\binom{n}{k}$	binomial coefficient (see Fig. M1645)
E.g.f.	exponential generating function $A_E(x) = \sum a_n x^n/n!$
Euler transform	of sequence $a_0, a_1, \ldots$ is sequence $b_1, b_2, \ldots$ where $\sum_{n=0}^{\infty} a_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{b_n}}$
$\exp x$	e^x
$F(n)$ or F_n	nth Fibonacci number (see Fig. M0692)

$p(n)$ or p_n	usually nth prime, occasionally nth partition number, but in latter case always identified as such
q	a prime or prime power
Ref	reference(s)
Rev.e.g.f.	reversion of exponential generating function
Rev.o.g.f.	reversion of ordinary generating function
w.r.t.	with respect to
XOR	logical "EXCLUSIVE OR", usually applied to binary representations of numbers
Λ_n	n-dimensional laminated lattice (see [SPLAG, Chap. 6])
$\mu(n)$	Möbius function (see M0011)
π	ratio of circumference of circle to diameter (see Fig. M2218)
$\prod$	a product, usually from 1 (or 0) to infinity, unless indicated otherwise
$\sigma(n)$	sum of divisors of n (see M2329)
$\sum$	a sum, usually from 0 (or 1) to infinity, unless indicated otherwise
τ	the golden ratio $(1+\sqrt{5})/2$ (see M4046)
$\phi(n)$	Euler totient function (see Fig. M0500)
$\uparrow$	exponentiation
!	factorial symbol: $0! = 1$, $n! = 1.2.3.\cdots.n$, $n \geq 1$ (see Fig. M4730)
#	number
$[x]$	largest integer not exceeding x
$\lceil x \rceil$	smallest integer not less than x

${}_mF_n$	hypergeometric series (see [Slat66]): ${}_mF_n([r_1, r_2, \ldots, r_m];\ [s_1, s_2, \ldots, s_n]; x) = \sum_{k=0}^{\infty} \frac{(r_1)_k (r_2)_k \cdots (r_m)_k}{(s_1)_k \cdots (s_n)_k} \frac{x^k}{k!}$, where $(r)_0 = 1$, $(r)_k = r(r+1)\cdots(r+k-1)$, for $k = 1, 2, \ldots$
f.c.c.	face-centered cubic lattice (see [SPLAG 112])
g.c.d.	greatest common divisor
G.f.	generating function, usually the ordinary generating function $A(x)$
h.c.p.	hexagonal close packing (see [SPLAG 113])
l.c.m.	least common multiple
Lgd.e.g.f.	logarithmic derivative of exponential generating function
Lgd.o.g.f.	logarithmic derivative of ordinary generating function
Möbius transformation	of sequence $a_1, a_2, \ldots$ is sequence $b_1, b_2, \ldots$, where $b_n = \sum_{d\mid n} \mu\left(\frac{n}{d}\right) a_d$, and $\mu(n)$ is the Möbius function M0011
multiplicative encoding	of a triangular array $\{t(n,k) \geq 0;\ n = 0, 1, \ldots$ and $0 \leq k \leq n\}$ is the sequence whose nth term is $\prod_{k=0}^{n} p_{k+1}^{t(n,k)}$, whose $p_1 = 2, p_2 = 3, \ldots$ are the primes
n	either a typical subscript , as in M0705: "$a(n) = a(n-1) + 2a(n-3)$", or a typical term in the sequence, as in M0641: "$6n-1$, $6n+1$ are twin primes"
O.g.f.	ordinary generating function $A(x)$
OR	logical "OR", usually applied to binary representations of numbers
p	a prime

Chapter 1

Description of the Book

It is the fate of those who toil at the lower employments of life, to be driven rather by the fear of evil, than attracted by the prospect of good; to be exposed to censure, without hope of praise; to be disgraced by miscarriage, or punished for neglect, where success would have been without applause, and diligence without reward.

Among these unhappy mortals is the writer of dictionaries; whom mankind have considered, not as the pupil, but the slave of science, the pionier of literature, doomed only to remove rubbish and clear obstructions from the paths of Learning and Genius, who press forward to conquest and glory, without bestowing a smile on the humble drudge that facilitates their progress. Every other authour may aspire to praise; the lexicographer can only hope to escape reproach, and even this negative recompense has yet been granted to very few.

Samuel Johnson, Preface to the "Dictionary," 1755

This epigraph, copied from the 1973 book, still applies!

1.1 Description of a Typical Entry

The main table is a list of about 5350 sequences of integers. A typical entry is:

M1484 `1,1,2,5,15,52,203,877,4140,21147,115975,678570, 4213597,27644437,190899322,1382958545,10480142147, 82864869804,682076806159,5832742205057`
Bell or exponential numbers: $a(n+1) = \Sigma a(k)C(n,k)$. See Fig M4981. Ref MOC 16 418 62. AMM 71 498 64. PSPM 19 172 71. GO71. [0,3; A0110, N0585]

$$\text{E.g.f.:}\ \exp(e^x - 1)$$

and consists of the following items:

M1484	The sequence identification number in this book
1, 1, 2, 5, 15, 52 , ...	The sequence itself
Bell or exponential numbers	Name or descriptive phrase
$a(n+1)$ $= \sum a(k)C(n,k)$	A recurrence: $a(n)$ is the nth term, $C(n,k)$ is a binomial coefficient, and the sum is over the natural range of the dummy variable, in this case over $k = 0, 1, \ldots, n$
See Fig M4981	Further information will be found in the figure accompanying sequence M4981
Ref	References
MOC 16 418 62	*Mathematics of Computation*, vol. 16, p. 418, 1962
AMM 71 498 64	*American Mathematical Monthly*, vol. 71, p. 498, 1964
	For other references, see the bibliography
0,3	The offset [inside square brackets]: the first number, 0, indicates that the first term given is $a(0)$, and the second number, 3, that the third term of the sequence is the first that exceeds 1 (the latter is used to determine the position of the sequence in the lexicographic order in the table)
A0110	Absolute identification number for the sequence
N0585	(If present) the identification number of the sequence in the 1973 book [HIS]
E.g.f.	Further information about the sequence (typically a generating function or recurrence) may be displayed following the sequence; in this case "E.g.f." indicates an exponential generating function — see Section 2.4.

We have attempted to give the simplest possible descriptions. In the descriptions, phrases such as "The number of" or "The number of distinct" have usually been omitted. Since there are often several ways to interpret "distinct", there may be more than one sequence with the same name. The principal sequences are described in detail, while less information is given about subsiduary ones. The indices are usually $0, 1, 2, \ldots$ or $1, 2, 3, \ldots$, or sometimes the primes $2, 3, 5, \ldots$. The first number in square brackets at the end of the description gives the initial index.

1.2 Arrangement of Table

The entries are arranged in lexicographic order, so that sequences beginning 2, 3, ... come before those beginning 2, 4, ..., etc. Any initial 0's and 1's are ignored when doing this.

1.3 Number of Terms Given

Whenever possible enough terms are given to fill two lines. If fewer terms are given it is because either no one knows the next term (as in sequences M0219, M0223, M0233, M0240, M0582, M5482, for example), or because although it would be straightforward to calculate the next term, no one has taken the trouble to do so (as in sequences M0115, M0163, M0406, M0686, M0704, M5485, etc.). We encourage every reader to pick a sequence, extend it, and send the results to the first author, whose address is given in Section 2.2. Of course some sequences are known to be hard to extend: see Fig. M2051. The current status of any sequence can be found via the email servers mentioned in Section 2.9.

1.4 References

To conserve space, journal references are extremely abbreviated. They usually give the exact page on which the sequence may be found, but neither the author nor the title of the article. To find out more the reader must go to a library; to get the most out of this book, it should be used in conjunction with a library.

Journal references usually give volume, page, and year, in that order. (See the example at beginning of this chapter.) Years after 1899 are abbreviated, by dropping the 19. Earlier years are not abbreviated. Sometimes to avoid ambiguity we use the more expanded form of: journal name (series number), volume number (issue number), page number, year.

References to books give volume (if any) and page. (See the example at the beginning of this chapter.)

The references do not attempt to give the discoverer of a sequence, but rather the most extensive table of the sequence that has been published.

In most cases the sequence will be found on the page cited. In some instances, however, for instance when we have not seen the article (if it is in an obscure conference proceedings, or more often because the sequence was taken from a pre-publication version) the reference is to the first page of the article. Our policy has been to include all interesting sequences, no matter how obscure the reference. In a few cases the reference does not describe the sequence itself but only a closely-related one.

1.5 What Sequences Are Included?

To be included, a sequence must satisfy the following rules (although exceptions have been made to each of them).

Rule 1. The sequence must consist of nonnegative integers.

Sequences with varying signs have been replaced by their absolute values.

Interesting sequences of fractions have been entered by numerators and denominators separately.

Arrays have been entered by rows, columns or diagonals, as appropriate, and in some cases by the multiplicative representation described in Fig. M1722.

Some sequences of real numbers have been replaced by their integer parts, others by the nearest integers.

The only genuine exceptions to Rule 1 are sequences such as M0728, M1551, which are integral for a considerable number of terms although eventually becoming nonintegral.

Rule 2. The sequence must be infinite.

Exceptions have been made to this rule for certain important number-theoretic sequences, such as Euler's idoneal (or suitable) numbers, M0476. Many sequences, such as the Mersemne primes, M0672, which are not yet known to be infinite, have been given the benefit of the doubt.

Rule 3. The first nontrivial term in the sequence, i.e. the first that exceeds 1, must be between 2 and 999.

The position of the sequence in the lexicographic order in the table is determined by the terms of the sequence beginning at this point.

The artificial sequences M0004 and M5487 mark the boundaries of the table.

Rule 3 implicitly excludes sequences consisting of only 0's and 1's. However, for technical reasons related to the sequence transformations discussed in Sect. 2.7, a few 0, 1 sequences have been included. They appear at the beginning of the main table.

Rule 4. The sequence must be well-defined and interesting. Ideally it should have appeared somewhere in the scientific literature, although there are many exceptions to this. Enough terms must be known to distinguish the sequence from its neighbors in the table, although one or two exceptions to this have been made for especially important sequences.

The selection has inevitably been subjective, but the goal has been to include a broad variety of sequences and as many as possible.

1.6 The Figures

The figures interspersed through the table give further information about certain sequences. Our aim, not fully achieved, was that taken together the figures and the table entries would give at least a brief description of the properties of the most important sequences. By combining the entry for the *subfactorial* or *rencontres* numbers, M1937, for instance, with the information from Fig. M1937, one can obtain a definition, exact formula, generating function and a recurrence for these numbers.

The figures serve two other purposes. One is to provide a short discussion of certain especially interesting families of sequences (such as "self-generating" sequences, Fig. M0436; famous hard sequences, Fig. M2051; or our favorite sequences, Fig. M2629).

The other is to display the most important *arrays* of numbers and the sequences to which they give rise — see Fig. M1645, for example, which describes some of the many sequences connected with the diagonals and even the rows of Pascal's triangle. These figures compensate to a certain extent for the fact that the book does not catalogue arrays of numbers.

Chapter 2

How to Handle a Strange Sequence

We begin with tests that can be done "by hand", then give tests needing a computer, and end by describing two on-line versions of the Encyclopedia that can be accessed via electronic mail.

2.1 How to See If a Sequence Is in the Table

Obtain as many terms of the sequence as possible. To look it up in the table, first omit all minus signs. Then find the first nontrivial term in the sequence, i.e. the first that exceeds 1. The terms beginning at this point determine where the sequence is placed in the lexicographic order in the table.

For example, to locate 1, 1, 1, 1, 1, $\underline{2}$, 1, 2, 3, 2, 3, ..., the underlined number is the first nontrivial term, so this sequence should be looked up in the table at 2, 1, 2, 3, 2, 3, ... (it is M0112).

For handling arrays, rationals or real numbers, see Section 1.5.

2.2 If the Sequence Is Not in the Table

- Try examining the differences between terms, as discussed in Section 2.5, and look for a pattern.
- Try transforming the sequence in some of the ways described in Section 2.7, and see if the transformed sequence is in the table.
- Try the further methods of attack that are mentioned in Sections 2.6 and 2.8.
- Send it by electronic mail to `superseeker@research.att.com`, as described in Section 2.9. This program automatically applies many of the

tests described in this chapter.

- If all these methods fail, and it seems that the sequence is neither in the Encyclopedia nor has a simple explanation, please send the sequence and anything that is known about it, including appropriate references, to the first author[1] for possible inclusion in the table.

2.3 Finding the Next Term

Suppose we are given the first few terms

$$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$$

of a sequence, and would like to find a rule or explanation for it. If nothing is known about the history or provenance of the sequence, nothing can be said, and any continuation is possible. (Any $n+1$ points can be fitted by an nth degree polynomial.)

But the sequences normally encountered, and those in this book, are distinguished in that they have been produced in some intelligent and systematic way. Occasionally such sequences have a simple explanation, and if so, the methods discussed in this chapter may help to find it. These methods can be divided roughly into two classes: those which look for a systematic way of generating the nth term a_n from the terms $a_0, \ldots, a_{n-1}$ before it, for instance by a recurrence such as $a_n = a_{n-1} + a_{n-2}$, i.e. methods which seek an *internal* explanation; and those which look for a systematic way of going from n to a_n, e.g. a_n is the number of divisors of n, or the number of trees with n nodes, or the nth prime number, i.e. methods which seek an *external* explanation. The methods in Sect. 2.5 and some of those in Sect. 2.6 are useful for attempting to discover internal explanations. External explanations are harder to find, although the transformations in Sect. 2.7 are of some help, in that they may reveal that the unknown sequence is a transformation of a sequence that has already been studied in some other context.

In spite of the warning given at the beginning of this section, in practice it is usually clear when the correct explanation for a sequence has been found. "Oh yes, of course!", one says.

There is an extensive literature dealing with the mathematical problems of defining the complexity of sequences. We will not discuss this subject here, but simply refer the reader to the literature: see for example Feder et al. [PGIT 38 1258 92],

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Fine [IC 16 331 70], [FI1], Martin-Lof [IC 9 602 66], Ziv [Capo90 366], Lempel and Ziv [PGIT 22 75 76], as well as a number of other papers by Ziv and his collaborators that have appeared in [PGIT].

2.4 Recurrences and Generating Functions

Let the sequence be $a_0, a_1, a_2, a_3, \ldots$. Is there a systematic way of getting the nth term a_n from the preceding terms $a_{n-1}, a_{n-2}, \ldots$? A rule for doing this, such as $a_n = a_{n-1}^2 - a_{n-2}$, is called a *recurrence*, and of course provides a method for getting as many terms of the sequence as desired.

When studying sequences and recurrences it is often convenient to represent the sequence by a power series such as

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

which is called its *(ordinary) generating function* (o.g.f. or simply g.f.), or

$$E(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots ,$$

its *exponential generating function* (or e.g.f.). (These are formal power series having the sequence as coefficients; questions of convergence will not concern us.)

For example, the sequence M2535: 1, 3, 6, 10, 15, ... of triangular numbers has

$$\begin{aligned} A(x) &= \frac{1}{(1-x)^3} , \\ E(x) &= \left(1 + 2x + \frac{x^2}{2}\right) e^x . \end{aligned}$$

Generating functions provide a very efficient way to represent sequences.

A great deal has been written about how generating functions can be used in mathematics: see for example Bender & Goldman [IUMJ 20 753 71], Bergeron, Labelle and Leroux [BLL94], Cameron [DM 75 89 89], Doubilet, Rota and Stanley [Rota75 83], Graham, Knuth and Patashnik [GKP], Harary and Palmer [HP73], Leroux and Miloudi [LeMi91], Riordan [R1], [RCI], Stanley [Stan86], Wilf [Wilf90]. (See also the very interesting work of Viennot [Vien83].)

Once a recurrence has been found for a sequence, techniques for solving it will be found in Batchelder [Batc27], Greene and Knuth [GK90], Levy and Lessman [LeLe59], Riordan [R1], and Wimp [Wimp84].

For example, consider M0692, the Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... These are generated by the recurrence $a_n = a_{n-1} + a_{n-2}$, and from this it is not difficult to obtain the generating function

$$1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots = \frac{1}{1 - x - x^2},$$

and the explicit formula for the nth term:

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right].$$

2.5 Analysis of Differences

This is the best method for analyzing a sequence "by hand". In favorable cases it will find a recurrence or an explicit formula for the nth term of a sequence, or at least it may suggest how to continue the sequence. It succeeds if the nth term is a polynomial in n, as well as in many other cases.

If the sequence is

$$a_0, a_1, a_2, a_3, a_4, \ldots ,$$

then its first differences are the numbers

$$\Delta a_0 = a_1 - a_0, \quad \Delta a_1 = a_2 - a_1, \quad \Delta a_2 = a_3 - a_2, \quad \ldots ,$$

its second differences are

$$\Delta^2 a_0 = \Delta a_1 - \Delta a_0, \quad \Delta^2 a_1 = \Delta a_2 - \Delta a_1, \quad \Delta^2 a_2 = \Delta a_3 - \Delta a_2, \quad \ldots ,$$

and so on. The 0th differences are the original sequence: $\Delta^0 a_0 = a_0$, $\Delta^0 a_1 = a_1$, $\Delta^0 a_2 = a_2, \ldots$; and the kth differences are

$$\Delta^k a_n = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n$$

or, in terms of the original sequence,

$$\Delta^k a_n = \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{n+k-i} . \tag{2.1}$$

Therefore if the differences of some order can be identified, Eq. (2.1) gives a recurrence for the sequence. Furthermore, if the differences a_m, Δa_m, $\Delta^2 a_m$, $\Delta^3 a_m, \ldots$ are known for some fixed value of m, then a formula for the nth term is given by

$$a_{n+m} = \sum_{k=0}^{n} \binom{n}{k} \Delta^k a_m . \tag{2.2}$$

The array of numbers

$$\begin{array}{ccccccccccc}
a_0 & & a_1 & & a_2 & & a_3 & & a_4 & & \cdots \\
& \Delta a_0 & & \Delta a_1 & & \Delta a_2 & & \Delta a_3 & & \cdots & \\
& & \Delta^2 a_0 & & \Delta^2 a_1 & & \Delta^2 a_2 & & \cdots & & \\
& & & \Delta^3 a_0 & & \Delta^3 a_1 & & \cdots & & & \\
& & & & \cdots & & & & & &
\end{array}$$

is called the *difference table of depth 1* for the sequence.

Example (i). M3818, the pentagonal numbers:

n	1		2		3		4		5		6		7		8
a_n	1		5		12		22		35		51		70		92
Δa_n		4		7		10		13		16		19		22	
$\Delta^2 a_n$			3		3		3		3		3		3		
$\Delta^3 a_n$				0		0		0		0		0			

Since $\Delta^2 a_n = 3$, $\Delta a_{n+1} - \Delta a_n = 3$, i.e. $a_{n+2} - 2a_{n+1} + a_n = 3$, which is a recurrence for the sequence. An explicit formula is obtained from Eq. (2.2) with $m = 1$:

$$a_{n+1} = 1 + 4\binom{n}{1} + 3\binom{n}{2} = 1 + 4n + 3\frac{n(n-1)}{2} = \frac{1}{2}(n+1)(3n+2) .$$

In general, if the rth differences are zero, a_n is a polynomial in n of degree $r-1$.

Example (ii). M3416, Eulerian numbers:

n	0		1		2		3		4		5		6		7
a_n	0		1		4		11		26		57		120		247
Δa_n		1		3		7		15		31		63		127	
$\Delta^2 a_n$			2		4		8		16		32		64		

Here $\Delta^2 a_n = 2^{n+1}$, $\Delta a_n = 2^{n+1} - 1$, and $a_n = 2^{n+1} - n - 2$. Equation (2.2) gives the same answer.

Example (iii). M1413, the Pell numbers:

$$
\begin{array}{r ccccccccccccc}
n & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 \\
a_n & 1 & & 2 & & 5 & & 12 & & 29 & & 70 & & 169 \\
\Delta a_n & & 1 & & 3 & & 7 & & 17 & & 41 & & 99 & \\
\Delta^2 a_n & & & 2 & & 4 & & 10 & & 24 & & 58 & & \\
\frac{1}{2}\Delta^2 a_n & & & 1 & & 2 & & 5 & & 12 & & 29 & &
\end{array}
$$

Since $\frac{1}{2}\Delta^2 a_n = a_n$, Eq. (2.1) gives the recurrence $a_{n+2} - 2a_{n+1} - a_n = 0$. Calculating further differences shows that $\Delta^k a_1 = 2^{[k/2]}$ and so Eq. (2.2) gives the formula

$$a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} 2^{[k/2]} .$$

If no pattern is visible in the difference table of depth 1, we may take the leading diagonal of that table to be the top row of a new difference table, the *difference table of depth* 2, and so on. For example, the difference table of depth 1 for

$$0, 2, 9, 31, 97, 291, 857$$

is

$$
\begin{array}{ccccccccccccc}
0 & & 2 & & 9 & & 31 & & 97 & & 291 & & 857 \\
& 2 & & 7 & & 22 & & 66 & & 194 & & 566 & \\
& & 5 & & 15 & & 44 & & 128 & & 372 & & \\
& & & 10 & & 29 & & 84 & & 244 & & & \\
& & & & 19 & & 55 & & 160 & & & & \\
& & & & & 36 & & 105 & & & & & \\
& & & & & & 69 & & & & & &
\end{array}
$$

No pattern is visible, so we compute the difference table of depth 2:

$$
\begin{array}{ccccccccccccc}
0 & & 2 & & 5 & & 10 & & 19 & & 36 & & 69 \\
& 2 & & 3 & & 5 & & 9 & & 17 & & 33 & \\
& & 1 & & 2 & & 4 & & 8 & & 16 & &
\end{array}
$$

Success! If we denote the sequence $0, 2, 5, \ldots$ by $b_0, b_1, b_2, \ldots$, then we see that $\Delta^2 b_n = 2^n$, $b_n = 2^n + n - 1$, and the original sequence is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (2^k + k - 1) .$$

In general, the relationship between the top row of a difference table

$$
\begin{array}{ccccccccccc}
b_0 = a_0 & & a_1 & & a_2 & & a_3 & & a_4 & & a_5 \\
& b_1 & & \cdot & & \cdot & & \cdot & & \cdot & \\
& & b_2 & & \cdot & & \cdot & & \cdot & & \\
& & & b_3 & & \cdot & & \cdot & & & \\
& & & & b_4 & & \cdot & & & &
\end{array}
$$

and the leading diagonal is given by

$$
a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \,, \quad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \,. \tag{2.3}
$$

2.6 Other Methods for Hand Analysis

• Try transforming the sequence in various ways — see Sect. 2.7. • Is the sequence close to a known sequence, such as the powers of 2? If so, try subtracting off the known sequence. For example, M3416 (again): 0, 1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, ... The last four numbers are close to powers of 2: 512, 1024, 2048, 4096; and then it is easy to find $a_n = 2^n - n - 1$.

• Is a simple recurrence such as $a_n = \alpha a_{n-1} + \beta a_{n-2}$ (where α, β are integers) likely? For this to happen, the ratio $\rho_n = a_{n+1}/a_n$ of successive terms must approach a constant as n increases. Use the first few values to determine α and β and then check if the remaining terms are generated correctly.

• If the ratio ρ_n has first differences which are approximately constant, this suggests a recurrence of the type $a_n = \alpha n a_{n-1} \cdots$. For example, M1783: 0, 1, 2, 7, 30, 157, 972, 6961, 56660, 516901, ... has successive ratios 2, 3.5, 4.29, 5.23, 6.19, 7.16, 8.14, 9.12, ... with differences approaching 1, suggesting $a_n = na_{n-1} + ?$. Subtracting na_{n-1} from a_n, we obtain the original sequence 0, 1, 2, 7, 30, 157, 972, ... again, so $a_n = na_{n-1} + a_{n-2}$.

This example illustrates the principle that whenever $\rho_n = a_{n+1}/a_n$ seems to be close to a recognizable sequence r_n, one should try to analyze the sequence $b_n = a_{n+1} - r_n a_n$.

• A recurrence of the form $a_n = na_{n-1} +$ (small term) can be identified by the fact that the 10th term is approximately 10 times the 9th. For example, M1937: 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, ..., $a_n = na_{n-1} + (-1)^n$.

• The recurrence $a_n = a_{n-1}^2 + \cdots$ is characterized by the fact that each term is about twice as long as the one before. For example, M0865: 2, 3, 7, 43, 1807, 3263443, 10650056950807, ..., and $a_n = a_{n-1}^2 - a_{n-1} + 1$.

• Does the sequence, or one obtained from it by some simple operation, have many factors? Consider the sequence 1, 5, 23, 119, 719, 5039, 40319, ... As it stands, the sequence cannot be factored, since 719 is prime, but the addition of 1 to all the terms gives the highly composite sequence $2, 6 = 2 \cdot 3$, $24 = 2 \cdot 3 \cdot 4$, $120 = 2 \cdot 3 \cdot 4 \cdot 5, \ldots$, which are the factorial numbers, M1675.

• The presence of only small primes may also suggest binomial coefficients. For example, M1459, the Catalan numbers: 1, 1, 2, 5, $14 = 2 \cdot 7$, $42 = 2 \cdot 3 \cdot 7$, $132 = 4 \cdot 3 \cdot 11, 429 = 3 \cdot 11 \cdot 13, 1430 = 2 \cdot 5 \cdot 11 \cdot 13, 4862 = 2 \cdot 11 \cdot 13 \cdot 17, \ldots$ and

$$a_n = \frac{1}{n+1}\binom{2n}{n}$$

(see Fig. M1459).

• Is there a pattern to the exponents in the prime factorization of the terms? E. g. M2050: $2 = 2^1$, $12 = 2^2 3^1$, $360 = 2^3 3^2 5^1$, etc.

• Sequences arising in number theory are sometimes *multiplicative*, i.e. have the property that $a_{mn} = a_m a_n$ whenever m and n have no common factor. For example, M0246: 1, 2, 2, 3, 2, 4, 2, 4, ..., the number of divisors of n.

• If the sequence is *two-valued*, i.e. takes on only two values X and Y (say), check if any of the six *characteristic sequences* can be recognized. The characteristic sequences, all essentially equivalent to the original sequence, are:

1. Replace X's and Y's by 1's and 2's
2. Replace X's and Y's by 2's and 1's
3. The sequence giving the positions of the X's
4. The sequence giving the positions of the Y's
5. The sequence of run lengths
6. The derivative sequence, i.e. the positions where the sequence changes

For example, the sequence

$$2, 2, 3, 3, 3, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, \ldots$$

contains runs of lengths

$$2, 3, 5, 7, 11, \ldots$$

which suggests the prime numbers as a possible explanation.

• Write the terms of the sequence in base 2, or base 3, ..., or base 8, and see if any pattern is visible. E.g. M2403: 0, 1, 3, 5, 7, 9, 15, 17, 21, ..., the binary expansion is a palindrome.

• If the terms in the sequence are all single digits, is it the decimal expansion of a recognizable constant? See Fig. M2218. If only digits in the range 0 to $b-1$ occur, is it the expansion of some constant in base b?

• Can anything be learned by considering the English words for the terms of the sequence? M1030 and M4780 are typical examples of sequences that can be explained in this way.

• There are a number of techniques for attempting to find a recurrence or generating function for a sequence. Most of these are best carried out by computer: see Sect. 2.8.

• **The quotient-difference algorithm.** One such method, however, can be carried out by hand. This procedure will succeed if the sequence satisfies a recurrence of the form

$$a_n = \sum_{i-1}^{r} c_i a_{n-i} \ , \tag{2.4}$$

where r and $c_1, \ldots, c_r$ are constants. The following description is due to Lunnon [Lunn74], who calls it the *quotient-difference algorithm*, since it is similar to a standard method in numerical analysis (cf. Gragg [SIAR 14 1 72], Henrici [Henr67], Jones and Thron [JoTh80]). The algorithm is also described by Conway and Guy [CoGu95]. Given a sequence $a_0, a_1, \ldots$, we form an array $\{S_{m,n}\}$ with $S_{0,n} = 1$ for all n, $S_{1,n} = a_n$, and in general

$$S_{mn} = \det \begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+m-1} \\ a_{n-1} & a_n & \cdots & a_{n+m-2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-m+1} & & \cdots & a_n \end{bmatrix} . \tag{2.5}$$

Any entry X in the array is related to its four neighbors

$$\begin{array}{ccc} & N & \\ W & X & E \\ & S & \end{array}$$

by the rule

$$X^2 = NS + EW \ , \tag{2.6}$$

and this can be used to build up much of the array, falling back on (2.5) when (2.6) is indeterminate. A recurrence of the form (2.4) holds if the $(r+1)$th row $S_{r+1,n}$ is identically zero.

For example, M2454: 1, 1, 1, 3, 5, 9, ... gives rise to the array

$$
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 3 & 5 & 9 & 17 & 31 & 57 \\
 & 0 & -2 & 4 & -2 & -4 & 10 & -8 & \\
 & & 4 & 4 & 4 & 4 & 4 & & \\
 & & & 0 & 0 & 0 & & &
\end{array}
$$

Row 4 is identically zero, and indeed

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \,.$$

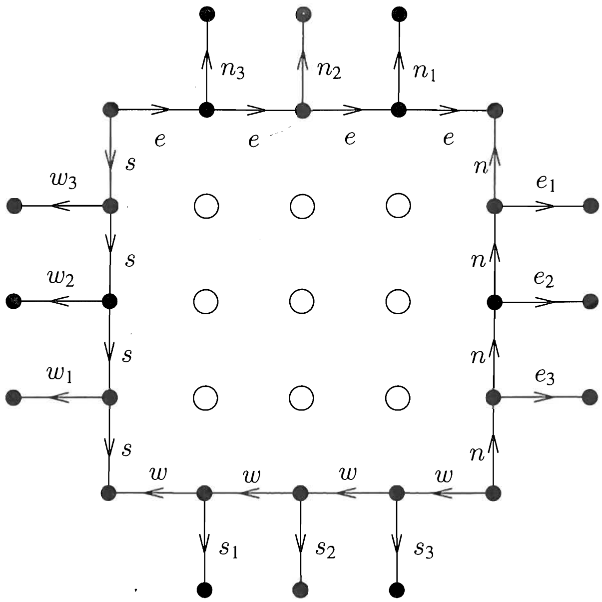
Zeros cause a problem in building the table, since then both sides of (2.6) vanish. Lunnon shows that the zeros always form square " windows", as illustrated in the following array for the sequence of Fibonacci numbers minus one (cf. M1056):

$$
\begin{array}{cccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & -4 & 1 & -2 & \boxed{0} & -1 & 0 & 0 & 1 & 2 & 4 & 7 & 12 & 20 \\
 & 12 & -7 & 4 & -2 & 1 & 0 & 0 & 1 & \boxed{0} & 2 & 1 & 4 & \\
 & & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & &
\end{array}
$$

There are simple rules for working past a window of zeros, found by J. H. Conway, and included here at his suggestion (see also [CoGu95]). To work past an isolated zero

$$
\begin{array}{ccccc}
 & & N' & & \\
 & & N & & \\
W' & W & 0 & E & E' \\
 & & S & & \\
 & & S' & &
\end{array}
$$

we use the rule that $N^2 S' + N' S^2 = W^2 E' + W' E^2$. To work around a larger window such as



we let $n, s, e, w, n_1, s_1, \ldots$ denote the ratios of the entries at the head and tail of the appropriate arrow. Then the rules are that

$$ns = \pm\, ew$$

($+$ for even-sized windows, $-$ for odd-sized), and

$$\begin{aligned}
\frac{s_1}{s} &= \frac{n_1}{n} - \frac{w_1}{w} + \frac{e_1}{e}\,, \\
\frac{s_2}{s} &= \frac{n_2}{n} + \frac{w_2}{w} - \frac{e_2}{e}\,, \\
\frac{s_3}{s} &= \frac{n_3}{n} - \frac{w_3}{w} + \frac{e_3}{e}\,,
\end{aligned}$$

etc.

However, if a computer is available, it is generally easier to use the `gfun` package (Sect. 2.8) than the quotient difference algorithm.

Getu et al. [SIAD 5 497 92] show that in some cases one can learn more by decomposing the matrix on the right-hand side of (2.5) into a product of lower triangular, diagonal, and upper triangular matrices.

• Is there any other way in which the nth term of the sequence could be produced from the preceding terms? Does the sequence fall into the class of what are loosely called *self-generating* sequences? A typical example is M0257: 1, 2, 2, 3, 3, 4, 4, 4, 5, ..., in which a_n is the number of times n appears in the sequence. See Figs. M0436, M0557 for further examples.

• Is this a Beatty sequence? If α and β are positive irrational numbers with $1/\alpha + 1/\beta = 1$, then the *Beatty sequences*

$$[\alpha], [2\alpha], [3\alpha], \ldots \quad \text{and} \quad [\beta], [2\beta], [3\beta], \ldots$$

together contain all the positive integers without repetition (see Fig. M1332). The following test for Beatty sequences is due to R. L. Graham. If $a_1, a_2, \ldots$ is a Beatty sequence, then the values of $a_1, \ldots, a_{n-1}$ determine a_n to within 1. Look at the sums $a_1 + a_{n-1}, a_2 + a_{n-2}, \ldots, a_{n-1} + a_1$. If all these sums have the same value, V say, then a_n must equal V or $V + 1$; but if they take on the two values V and $V + 1$, and no others, then a_n must equal $V + 1$. If anything else happens, it is not a Beatty sequence. For example, in the Beatty sequence M2322: 1, 3, 4, 6, 8, 9, ..., we have $a_1 + a_1 = 2$ so a_2 must be 2 or 3 (it is 3); $a_1 + a_2 = 4$ so a_3 must be 4 or 5 (it is 4); $a_1 + a_3 = 5$ and $a_2 + a_2 = 6$, so a_4 must be 6 (it is); and so on.

2.7 Transformations of Sequences

One of the most powerful techniques for investigating a strange sequence is to transform it in some way and see if the resulting sequence is either in the table or can be otherwise identified. (A more elaborate procedure, at present prohibitively expensive, would apply these transformations both to the unknown sequence and to all the sequences in the table, and then look for a match between the two lists.)

For example, the sequence 1, 4, 5, 11, 10, 20, 14, 27, 24, 34, ... (of no special interest, invented simply to illustrate this point), is not in the table. But the Möbius transform of it (defined below) is 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, ... which is M2336, the sequence of numbers that are of the form $x^2 + xy + y^2$.

This section describes some of the principal transformations that can be applied. Although any single transformation can be performed by hand, a thorough investigation using these methods is best carried out by computer. The program `superseeker` described in Sect. 2.9 tries many such transformation.

Our notation is that $a_0, a_1, a_2, \ldots$ is the unknown sequence, and $A(x)$ and $A_E(x)$ are its ordinary and exponential generating functions; $b_0, b_1, b_2, \ldots$ is the transformed sequence with o.g.f. $B(x)$ and e.g.f. $B_E(x)$.

We begin with some elementary transformations. The reader will easily invent many others of a similar nature. (`Superseeker` actually tries over 100 such transformations.)

• Translations: $b_n = a_n + c$; $b_n = a_n + n + c$; $b_n = a_n - n + c$; where c is $-3, -2, -1, 0, 1, 2,$ or 3.

• Rescaling: $b_n = 2a_n$; $b_n = 3a_n$; $b_n = a_n$ divided by the g.c.d. of all the a_i's; the same after deleting a_0; the same after deleting a_0 and a_1; $b_n = a_n/n!$ (if integral). If all a_n are odd, set $b_n = (a_n - 1)/2$.

• Differences: $b_n = \Delta a_n$; $b_n = \Delta^2 a_n$; etc. If a_n divides a_{n+1} for all n, set $b_n = a_{n+1}/a_n$.

• Sums of adjacent terms: $b_n = a_n + a_{n-1}$; $b_n = a_n + a_{n-2}$.

• Bisections: $b_n = a_{2n}$; $b_n = a_{2n+1}$; trisections: $b_n = a_{3n}$; $b_n = a_{3n+1}$, $b_n = a_{3n+2}$, etc.

• Reciprocal of generating function: $B(x) = 1/A(x)$. For the combinatorial interpretation of b_n in this case see Cameron [DM 75 91 89].

• Other operations on $A(x)$: $B(x) = A(x)^2$; $1/A(x)^2$; $A(x)/(1-x)$ [so that $b_n = \sum\limits_{k \le n} a_k$]; $A(x)/(1-x)^2$; etc.

• Similar operations on $A_E(x)$: $B_E(x) = A_E(x)^2$; $1/A_E(x)$; etc.

• Complementary sequences. Those numbers not in the original sequence. Also $b_n = n - a_n$; $b_n = \binom{n}{2} - a_n$.

The following transformations are rather more interesting.

• **Exponential and logarithmic transforms.** Several versions are possible, but the usual one transforms $a_1, a_2, a_3, \ldots$ into $b_1, b_2, b_3, \ldots$ via

$$1 + \sum_{n=1}^{\infty} \frac{b_n x^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{a_n x^n}{n!}\right), \tag{2.7}$$

i.e.

$$1 + B_E(x) = \exp A_E(x) . \tag{2.8}$$

There is a combinatorial interpretation. For example, if a_n is the number of connected labeled graphs on n nodes, M3671, then $b_n = 2^{\binom{n}{2}}$, M1897, is the total number of connected or disconnected labeled graphs on n nodes. More generally, if a_n is the number of connected labeled graphs with a certain property, then b_n is the total number of labeled graphs with that property. Eq. (2.7) is Riddell's formula for labeled graphs (Harary and Palmer [HP73 8]).

Of course the inverse transformation is

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{n!} = \log\left(1 + \sum_{n=1}^{\infty} \frac{b_n x^n}{n!}\right) . \tag{2.9}$$

In this situation we say that b_1, $b_2, \ldots$ is the *exponential transform* of $a_1, a_2, \ldots$, and that $a_1, a_2, \ldots$ is the *logarithmic transform* of $b_1, b_2, \ldots$.

• **The Euler transform.** For unlabeled graphs a different pair of transformations applies. If two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are related by

$$1 + \sum_{n=1}^{\infty} b_n x^n = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)^{a_i}} , \tag{2.10}$$

or equivalently

$$1 + B(x) = \exp\left(\sum_{k=1}^{\infty} \frac{A(x^k)}{k}\right) , \tag{2.11}$$

then we say that $\{b_n\}$ is the *Euler* transform of $\{a_n\}$, and that $\{a_n\}$ is the *inverse Euler* transform of $\{b_n\}$.

Calculations are facilitated by introducing an intermediate sequence $c_1, c_2, \ldots$ defined by

$$c_n = \sum_{d|n} d a_d , \tag{2.12}$$

or

$$c_n = n b_n - \sum_{k=1}^{n-1} c_k b_{n-k} , \tag{2.13}$$

with

$$a_n = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) c_d , \tag{2.14}$$

where μ is the Möbius function (see M0011 and Fig. M0500). Using these formula $\{b_n\}$ can be obtained from $\{a_n\}$, or vice versa. The c_n have generating function

$$\log(1 + B(x)) = \sum_{n=1}^{\infty} c_n \frac{x^n}{n} . \tag{2.15}$$

There are many applications of this pair of transforms. In graph theory, if a_n is the number of connected, unlabeled graphs with some property, then b_n is the total number of graphs (connected or not) with the same property. In this context (2.11) is sometimes called Riddell's formula for unlabeled graphs (cf. Cadogan [JCT B11 193 71], Harary and Palmer [HP73 90]).

For example, if a_n $(n \geq 1)$ is the number of connected unlabeled graphs with n nodes, M1657: $1, 1, 2, 6, 21, \ldots$, then $b_n (n \geq 1)$ is the total number of unlabeled graphs with n nodes, M1253: 1, 2, 4, 11, ... The intermediate sequence c_n is M2691: $1, 3, 7, 27, \ldots$.

There are also number-theoretic applications: b_n is the number of partitions of n into integer parts of which there are a_1 different types of parts of size 1, a_2 of size 2, and so on. E.g. if all $a_n = 1$, then b_n is simply the number of partitions of n into integer parts (M0663). If $a_n = 1$ when n is a prime and 0 when n is composite, b_n is the number of partitions of n into prime parts (M0265). An important example of the $\{b_n\}$ sequence is M0266, which arises in connection with the Rogers-Ramanujan identities — see Andrews [Andr85], Andrews and Baxter [AMM 96 403 89]. Andrews [Andr85] discusses a number of other number-theoretic applications, and Cameron [DM 75 89 89] gives further applications in other parts of mathematics.

• **The Möbius transform.** If sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are related by

$$b_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) a_d \, , \tag{2.16}$$

$$a_n = \sum_{d|n} b_d \, , \tag{2.17}$$

where the summations are taken over all positive integers d that divide n, we say that $\{b_n\}$ is the *Möbius* transform of $\{a_n\}$, and that $\{a_n\}$ is the *inverse Möbius* (or *sum-of-divisors*) transform of $\{b_n\}$. Equations (2.16), (2.17) are called the *Möbius inversion formulae*. (The sequences in (2.12) and (2.14) are related in this way.) Two equivalent formulations are

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} b_n \frac{x^n}{1 - x^n} \, , \tag{2.18}$$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{b_n}{n^s} \, , \tag{2.19}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}} \tag{2.20}$$

is the Riemann zeta function.

Again there are many applications. For combinatorial applications see Rota [ZFW 2 340 64] (as well as several other papers reprinted in [GeRo87]), Bender and Goldman [AMM 82 789 75], and Stanley [Stan86]. For number-theoretic applications see for example Hardy and Wright [HWI §17.10] — the right-hand side of (2.18) is called a *Lambert series*.

Examples. (i) If $b_n = 1, 1, 1, \ldots$, $a_n =$ number of divisors of n (M0246). (ii) If $b_n = 1, 0, 0, \ldots$, $a_n =$ Möbius function (M0011). (iii) If $b_n = n$, $a_n =$ Euler totient function (M0299). (iv) If $b_{2n} = 0$, $b_{2n+1} = (-1)^n 4$, then $a_n =$ number of ways of writing n as a sum of two squares (M3218).

• **The binomial transform.** If $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are related as in Eq. (2.3), we say that $\{a_n\}$ is the *binomial* transform of $\{b_n\}$, and that $\{b_n\}$ is the *inverse binomial* transform of $\{a_n\}$. Equivalently, the exponential generating functions are related by

$$A_E(x) = e^x B_E(x) \ . \tag{2.21}$$

As we saw in Sect. 2.5, these transformations arise in studying the differences of a sequence. The leading diagonal of the difference table of a sequence is the inverse binomial transform of the sequence.

Examples. If $a_n = 3^n$, $b_n = 2^n$, and more generally, if $a_n = k^n$, $b_n = (k-1)^n$.

The Bell numbers 1, 1, 2, 5, 15, 52, ... (M1484) are distinguished by the property that they are shifted one place by the binomial transform: $a_n = b_{n+1}$ [BeS194].

• **Reversion of series.** Given a sequence $a_1, a_2, a_3, \ldots$ we can form a generating function

$$y = x(1 + a_1 x + a_2 x^2 + \cdots) \ , \tag{2.22}$$

and by expressing x in terms of y obtain a new sequence $b_1, b_2, b_3, \ldots$ by writing

$$x = y(1 - b_1 y - b_2 y^2 - \cdots) \ . \tag{2.23}$$

This process is called *reversion of series*, and explicit formulae expressing b_n in terms of $a_1, \ldots, a_n$ can be found for example in [AS1 16], [RCI 149], [TMJ 2 73 92]. This transformation is its own inverse. For example, if the a_n are the Fibonacci numbers 1, 2, 3, 5, 8, ... (M0692), the b_n are 1, 2, 5, 15, 51, 188, ... (M1480). It is amusing that the latter sequence is also the binomial transform of the Catalan numbers (M1459). An alternative version of this transformation is: given $a_0 = 1$, $a_1, \ldots$ we set $y = \sum_{i=0}^{\infty} a_i x^{i+1}$, whose reversion is $x = \sum_{i=0}^{\infty} b_i x^{i+1}$, producing the transformed sequence $b_0 = 1, b_1, \ldots$.

• **Other transforms.** A pair of transforms of the form

$$a_n = \sum C_{n,k} b_k, \quad b_n = \sum D_{n,k} a_k$$

can be defined whenever we find integer arrays $\{C_{n,k}\}$ and $\{D_{n,k}\}$ satisfying the orthogonality relation

$$\sum C_{m,k} D_{k,n} = \begin{cases} 1 & m = n, \\ 0 & m \neq n\,. \end{cases}$$

Riordan's book [RCI] gives many such examples, including transforms that are based on Chebyshev and Legendre polynomials.

We conclude by mentioning that the pair of transforms based on Stirling numbers seems to be worth investigating further, particularly in the context of enumerating permutations. In this case we have

$$a_n = \sum_{k=0}^{n} s(n,k) b_k, \quad b_n = \sum_{k=0}^{n} S(n,k) a_k\,, \tag{2.24}$$

where the coefficients are Stirling numbers of the first and second kinds, respectively (see Figs. M4730, M4981; also [R1 48], [RCI 90], [GKP 252], [BeS194]).

2.8 Methods for Computer Investigation of Sequences

As we have already mentioned, a thorough investigation of the transformations of a sequence described in the previous section is best done by computer.

• `Gfun`. At the heart of the following techniques is an algorithm of Cabay and Choi [SIAC 15 243 86] that uses Padé approximations to take a truncated power series

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \tag{2.25}$$

with rational coefficients, and determines a rational function $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with rational coefficients, whose Taylor series expansion agrees with (2.25) and in which $\deg p + \deg q$ is minimized. If $\deg p + \deg q < n-2$, we say this is a "good" representation of (2.25) (for then $p(x)/q(x)$ contains fewer constants than the original series).

The Cabay-Choi algorithm is incorporated in the Maple `convert/ratpoly` procedure. Bergeron and Plouffe [EXPM 1 307 92] observed that this provides an efficient way to search for a wide class of generating functions for sequences. Given a sequence $a_0, a_1, \ldots, a_{n-1}$, one can form the o.g.f. $A(x)$ and e.g.f. $A_E(x)$, and see if either have a "good" rational representation. If not, one can try again with the *logarithmic derivates* $A'(x)/A(x)$ and $A'_E(x)/A_E(x)$, and with many other transformed generating functions. In this way Bergeron and S.P. were able to find generating functions such as $1/(2-e^x)$ for M2952.

This work was carried much further by S.P. in his thesis [Plou92], which gives over 1000 generating functions, recurrences and formulae for the 4500 sequences in a 1991 version of the present table. Some of these are immediate, others can be proved with difficulty, but a considerable number are still only conjectural. The simplest of these (but not the conjectural ones) have now been incorporated in the table. To have included the rest, which are usually quite complicated, would have greatly increased the length of this book.

The `gfun` Maple package of Salvy and Zimmermann [SaZi94] incorporates and greatly extends the ideas of Bergeron and S.P. With `gfun`, one can (among many other things) check very easily:

(a) whether there is a "good" rational function representation for the o.g.f. or e.g.f. of a sequence, or for their logarithmic derivatives, or their reversions;

(b) whether the generating function $y(x)$ of any of these types satisfies a polynomial equation or a linear differential equation with polynomial coefficients;

(c) whether the coefficients of any of these generating functions satisfy a linear recurrence with polynomial coefficients;

and many other things. The package contains a number of commands that make it easy to manipulate sequences and power series and to convert between different types. The `superseeker` program described in Section 2.9 makes good use of `gfun`.

• **Look for sequences in the table that are close to the unknown sequence.** There are a number of ways to do this. Let $a = a_0, a_1, \ldots, a_{n-1}$ be the unknown sequence, and $b = b_0, b_1, \ldots, b_{m-1}$ a typical sequence in the table. We truncate the longer sequence so they both contain the same number of terms, n. Then we may ask:

(a) Which sequences in the table are closest in L_1 norm, i.e. minimize

$$\sum_{i=0}^{n-1} |a_i - b_i| \ ?$$

(b) Is there a sequence in the table such that

$$|a_i - b_i| \leq 1 \quad \text{for all} \quad i \ ?$$

Or for which $|a_i - b_i|$ is a constant sequence?

(c) Which sequences in the table are closest in Hamming distance? (Write a and b as strings of decimal digits and spaces, and count the places where they differ.)

(d) Which sequences in the table are most closely *correlated* with the unknown sequence? I.e., which maximize the squared correlation coefficient

$$r^2 = \frac{1}{(n-1)^2 s_a^2 s_b^2} \left(\sum_{i=0}^{n-1} (a_i - \bar{a})(b_i - \bar{b}) \right)^2 ,$$

where

$$\bar{a} = \frac{1}{n} \sum_{i=0}^{n-1} a_i, \quad s_a^2 = \frac{1}{n-1} \sum_{i=0}^{n-1} (a_i - \bar{a})^2$$

are the mean and variance of a, with similar definitions for $\bar{b}$ and s_b^2.

Notes: Among other things, (a) will detect small errors in calculation; (b) will detect sequences whose definition differs by a constant from one in the table; (c) will detect typing errors; (d) is the most time-consuming of these tests, and will detect a sequence of the form $a = pb + q$, where b is in the table and p and q are constants.

Another possible test of this type is to see if a is a subsequence of some sequence in the table, but we have not found this useful.

The remaining tests in this section are more speculative. However, once in a while they may find an explanation for a sequence that has not succumbed to any other test.

• **Apply the Berlekamp-Massey or Reed-Sloane algorithms.** Suppose the sequence takes on only a small number of different values, e.g. $\{0, 1, 2, 3\}$. By regarding the values as the elements of a finite field (the Galois field $GF(4)$ would be appropriate in this case) we may think of the sequence as a sequence from this field. The Berlekamp-Massey algorithm is an efficient procedure for finding the shortest linear recurrence with coefficients from the field that will generate the sequence — see Berlekamp [Be68 Chap. 7] and Massey [PGIT 15 122 69].

(Other references that discuss this extremely useful algorithm are Dickinson et al. [PGAC 19 31 74], Berlekamp et al. [UM 5 305 74], Mills [MOC 29 173 75], Gustavson [IBMJ 20 204 76], McEliece [McEl77], MacWilliams and Sloane [MS78 Chap. 9], and Brent et al. [JAlgo 1 259 80].) This algorithm would discover for example that the sequence

$$0, 1, 2, 1, 3, 0, 3, 0, 1, 3, 3, 2, 3, 3, 3, 1, 2, 0, 1, 1, 0, 0, \ldots$$

is generated by the linear recurrence

$$a_n = \omega(a_{n-1} + a_{n-2} + a_{n-3})$$

over $GF(4)$, where we take $GF(4)$ to consist of the elements $\{0, 1, \omega, \omega^2\}$, with $\omega^2 = \omega + 1$, and write 2 for ω, 3 for ω^2. The Reed-Sloane algorithm [SIAC 14 505 85] is an extension of this algorithm which applies when the terms of the sequences are integers modulo m, for some given modulus m. For example, this algorithm would discover that the sequence

$$1, 2, 4, 3, 1, 3, 6, 7, 4, 4, 1, 5, 3, 0, 5, 6, \ldots$$

is produced by the recurrence

$$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} \pmod 8 .$$

• **Apply a data compression algorithm.** Feed the sequence to a data compression algorithm, such as the Ziv-Lempel algorithm as implemented in the Unix commands `compress` or `gzip`.

If the sequence is compressed to a much greater extent than a comparable random sequence of the same length would be, there is some structure present that can be recovered by examining the compression algorithm (see for example [BCW90]).

For example, `gzip` compresses M0001 from 150 characters to 36 characters, whereas a random binary sequence of the same length typically is compressed only to 60 bits. So if a 150-character binary sequence is compressed to (say) 45 bits or less, one can be sure it has some concealed structure.

It would be worth running this test on any stubborn sequence which contains only a limited set of symbols. By experimenting with random sequences of the same length and containing the same symbols, one can determine their average compressibility. If the stubborn sequence is compressed to a greater degree than this then it has some hidden structure.

• **Compute the Fourier transform of the sequence.** An article by Loxton [Loxt89] demonstrates that the Fourier transform of a sequence can reveal much about how it is generated. This is a topic that deserves further investigation.

2.9 The On-Line Versions of the Encyclopedia

There are two on-line versions of the Encyclopedia that can be accessed via electronic mail. The first is a simple look-up service, while the second tries very hard to find an explanation for a sequence. Both make use of the latest and most up-to-date version of the main table.

To use the simple look-up service, send email to

`sequences@research.att.com`

containing lines of the form

lookup 5 14 42 132 429

There may be up to five such lines in a message. The program will automatically inform you of the first seven sequences in the table that match each line. If there are no "lookup" lines, you will be sent an instruction file.

Notes. When submitting a sequence, separate the terms by spaces (not commas). It may be advisable to omit the initial term, since there are often different opinions about how a sequence should begin. (Does one start counting graphs, say, at 0 nodes or at 1 node? Do the Lucas numbers begin 1, 3, 4, 7, 11, ... or 2, 1, 3, 4, 7, 11, ...?) Omit all minus signs, since they have been omitted from the table. If you receive seven matches to a sequence, try again giving more terms. For more details, see [Sloa94].

The second server not only looks up the sequence in the table, it also tries hard to find an explanation for it, using many of the tricks described in this chapter (and possibly others — at the time of writing the program is still being expanded). To use this more powerful program, send email to

`superseeker@research.att.com`

containing a line of the form

lookup 1 2 4 6 10 14 20 26 36 46 60 74 94 114 140 166

The program will apply many tests, and report any potentially useful information it discovers.

Notes. The word "lookup" should appear only once in the message. The terms of the sequence should be separated by spaces (not commas). For this program the sequence should be given from the beginning. Minus signs *should be included*, since most of the programs will make use of them. If possible, give from 10 to 20 terms. If you receive seven matches from the table, try again giving more terms.

2.10 The Floppy Disk

A floppy disk containing every sequence in the table (although not their descriptions) is available from the publisher. Please contact Academic Press at

1–800–321–5068 for information regarding the floppy disk to accompany *The Encyclopedia of Integer Sequences*. Please indicate desired format by referring to the ISBN for Macintosh (0-12-558631-0) or for IBM/MSDOS (0-12-558632-9).

The disk contains a line such as

```
M[1916] := [A6226, 1, 2, 9, 18, 118]:
```

for each sequence. The first number gives the sequence number in this book, the second gives the absolute identification number for the sequence, and the remaining numbers are the sequence itself.

This disk will enable readers to study the sequences in their own computers. Of course the book will still be needed for the descriptions of the sequences and the references.

Chapter 3

Further Topics

3.1 Applications

We begin by describing some typical ways in which the 1973 book [HIS] has been used, as well as some applications of the sequence servers mentioned in Section 2.9. (Even though at the time of writing the latter have been in existence for only a few months, there have already been some interesting applications). It is to be expected that the present book will find similar applications.

The most important way the table is used is in discovering whether someone has already worked on your problem. Discrete mathematics has grown exponentially over the last thirty years, and so there is a good chance that someone has already looked at the same problem, or an equivalent one. In this respect the book serves as an index, or field guide, to a broad spectrum of mathematics. If the answers to the first few special cases of a problem can be described by integers, and someone has considered the problem worth studying, there is a good chance you will find the sequence of numbers in this book. Of course if not, and if `superseeker` can't do anything with it, you should send in the sequence so that it can be added to the table — see Sect. 2.2 for instructions. Apart from anything else, this stakes out your claim to the problem! But, more important, you will be performing a service to the scientific community.

As with any dictionary (and as predicted by the epigraph to Chapter 1), most such successful uses go unrecorded. The reader simply stops working on the problem, as soon as he or she has been pointed to the appropriate place in the literature.

In many cases the book has led to mathematical discoveries. The following stories are typical.

• R. L. Graham and D. H. Lehmer were investigating the permanent P_n of Schur's matrix, the $n \times n$ matrix (α^{jk}), $0 \le j, k \le n-1$, where $\alpha = e^{2\pi i/n}$, and found that the initial values $P_1, P_3, P_5, \ldots$ were

$$1, -3, -5, -105, 81, \ldots$$

(P_n is 0 if n is even). As it happened, this sequence (M2509) was in the Supplement [Supp74] to the 1973 book, and N. J. A. S. was able to refer Graham and Lehmer to an earlier paper by D. H. Lehmer, where the same sequence had arisen! This provided an unexpected connection with circulant matrices [JAuMS A21 496 76].

• Extract from a letter about the 1973 book: "After reading about your book in *Scientific American*, I ordered a copy. Several of my friends looked at the book and stated they thought it was interesting but doubted its usefulness. A few days later I was attempting to determine the number of spanning trees on an n by m lattice. In working out the 2 by m case, I determined the first numbers in the sequence to be 1, 4, 15, 56. Noticing that both sequences No. 1420 and 1421 started this way, I worked out another term, 209; thus sequence No. 1420 seemed to fit. After much thought I was able to establish a complicated recursion relationship which I was later able to show was equivalent to the recursion you gave for No. 1420. ... In closing I would like to say that your book has already proved to be worthwhile to me since it provided guidelines for organizing my thoughts on this problem and suggested a hypothesis for the next term of the sequence. I'm sold!" (Alamogordo, New Mexico).

• While investigating a problem arising from cellular radio, Mira Bernstein, Paul Wright and N. J. A. S. were led to consider the number of sublattices of index n of the planar hexagonal lattice. For $n = 1, 2, 3, \ldots$ they calculated that these numbers were $1, 1, 2, 3, 2, 3, 3, 5, \ldots$. To their surprise, the table revealed that this sequence, M0420, had arisen in 1973 in an apparently totally different context, that of enumerating maps on a torus (Altshuler [DM 4 201 73]), and supplied a recurrence that they had overlooked. (However, it is only fair to add that the earlier paper did not find the elegant exact formula for the nth term that is given in [BSW94]. There is also an error in the values given in the earlier paper: $\chi(16)$ should be 9, not 16.)

• C. L. Mallows was interested in determining the number of statistical models with n factors, in particular linear hierarchical models that allow 2-way interactions. For $n = 1, 2, \ldots$ he found the numbers of such models to be

$$2, 4, 8, 19, 53, 209 .$$

This sequence was not at that time in the table, but `superseeker` (see Sect. 2.9) pointed out that these numbers agreed with the partial sums of M1253, the number of graphs on n nodes. With this hint, Mallows was instantly able to show that this explained his sequence (which is now M1153).

• R. K. Guy and W. O. J. Moser [GuMo94] report a successful application of `superseeker` in finding a recurrence for the number of subsequences of $[1, 2, \ldots, n]$ in which every odd number has at least one even neighbor. The first try with the program was unsuccessful, because of an error in one of their terms, but when the corrected sequence

$$1, 1, 3, 5, 11, 17, 39, 61, 139, \ldots$$

(now M2480) was submitted, `superseeker` used `gfun` to find the elegant generating function

$$\frac{1 + x + 2x^3}{1 - 3x^2 - 2x^4} .$$

• Inspection of the log file for the sequence servers on March 28, 1994 shows that at least one high-school student used the program to identify a sequence (M2638) for her homework.

Another important application of the book is to suggest possible connections between sequences arising in different areas, as in the Mallows story above. Here is a typical (although ultimately unsuccessful) example.

• The dimensions of the spaces of primitive Vassiliev knot invariants of orders $1, \ldots, 9$ form the sequence

$$1, 1, 1, 2, 3, 5, 8, 12, 18$$

the next term being presently unknown (see Birman [BAMS 28 281 93], Bar-Natan [BarN94]). This sequence coincides with the beginning of M0687, which gives the number of ways of arranging n pennies in rows of contiguous pennies, each touching two in the row below. Alas, further investigation by D. Bar-Natan has shown that next term in the former sequence is at least 27, and so these sequences are in fact *not* the same.

• As already mentioned in Sect. 2.8, S.P.'s thesis [Plou92] contains many conjectures about possible generating functions. For example, M2401, the size of the smallest square into which one can pack squares of sizes $1, 2, \ldots, n$, appeared to have generating function

$$(1 - z)^{-3}(1 - z^2) \prod_{m=4}^{\infty} (1 - z^{2m+1})(1 - z^{2m})^{-1} ,$$

which agreed with the 17 values known at the time [UPG D5]. This prompted R. K. Guy [rkg] to calculate some further terms, and to show that in fact this generating function is not correct. At present no general formula is known for this sequence.

For an example of a conjectured generating function (for M2306) that turned out to be correct, see Allouche et al. [AABB].

3.2 History

I[1] started collecting sequences in 1965 when I was a graduate student at Cornell University. I had run across several sequences whose asymptotic behavior I needed to determine, so I was hoping to find recurrences for them. Although John Riordan's book [R1] was full of sequences, the ones I was interested in did not seem to be there. Or were they? It was hard to tell, certainly some very similar sequences were mentioned. So I started collecting sequences on punched cards. Almost thirty years later, the collection is still growing (although it is no longer on punched cards.)

Over the course of several years I systematically searched through all the books and journals in the Cornell mathematics library, and then the Bell Labs library, when I joined the Labs in 1969. A visit to Brown University, with its marvelous collection of older mathematics books and journals, filled in many gaps. I never did find the sequences I was originally looking for, although of course they are now in the table (M4558 was the one I was most interested in: $0, 1, 8, 78, 944, 13800, \ldots$ a very familiar sequence! It essentially gives the average height of a rooted labeled tree.)

The first book [HIS] was finally published by Academic Press in 1973, and a supplement [Supp74] was issued a year later. Over the next fifteen years new material poured in, and by 1990 over a cubic meter of letters, articles, preprints, postcards, etc., had accumulated in my office. I made one attempt to revise the book in 1980, with the help of two summer students, Bob Hinman and Tray Peck, and managed to transfer the 1973 table from punched cards to magnetic disk, and started processing the new material. But at the end of that summer other projects intervened (cf. [MS78], [SPLAG]). Ten years later the amount of material waiting to be processed was overwhelming.

Fortunately S.P. wrote to me in 1991, offering to help with a new edition, and this provided the stimulus that, four years later, has produced the new book. It very nearly never happened!

3.3 Differences from the 1973 Book

• Size: There are now 5488 sequences, compared with 2372 in [HIS].

• Format: In [HIS], every sequence was normalized so as to begin $1, n$, with $2 \leq n \leq 999$, an initial 1 being added as a marker if necessary. Now the sequence can begin in any way, subject only to Rule 3 of Sect. 1.5.

[1]The first person seems appropriate here (N.J.A.S.).

• The descriptions are much more informative. Many generating functions have been included. One of the benefits of the transition from punched cards to magnetic disk has been an enlarged character set. Before, only upper case letters could be used; now, all standard mathematical symbols are available.

• All known errors in [HIS] have been corrected. In almost every case these were errors in the source material, not in transcription. Some erroneous or worthless sequences have been omitted.

• There is also a technical change. In the older mathematical literature 1 was regarded as a prime number, whereas today it is not. This has necessitated changes to a few sequences. M3352 for example now begins $4, 9, 11, \ldots$ rather than $2, 4, 9, 11, \ldots$ as in [HIS].

3.4 Future Plans

• The table should be modified so as to include minus signs. Unfortunately to do this thoroughly would require re-examining thousands of sequences, and this book has already been delayed long enough.

• It would be nice to have a series of essays, one for each family of sequences (Boolean functions, partitions, graphs, lattices, etc.), showing how the sequences are related to each other and which are fundamental. This would clarify the sequences that one should concentrate on when looking for generating functions, finding more terms, and so on. The late Victor Meally spent a great deal of time on such a project, and every square centimeter of his copy of [HIS], now in the Strens collection of the University of Calgary library, is annotated with cross-references between sequences, tables, diagrams, and so on — in other words a greatly expanded version of the Figures in the present book. It would be worthwhile doing this in a systematic way. Such commentaries could easily fill a companion volume.

• It would also be useful to classify the sequences into various categories, a multiple classification that would indicate:

- subject (graphs, partitions, etc.),
- type (enumerative, number-theoretic, dependent on base 10 representation, frivolous, etc.), and
- method of generation (ranging from "explicit formula", "recurrence", etc., to "the next term not known").

It is surprisingly difficult to give precise definitions for some of these classes —

there are explicit formulae for the nth prime, for instance, and the most intractable enumeration problem can be encoded into a recurrence if one defines enough variables (see for example [JCT 5 135 68]).

There are however a number of mathematically well-defined classes of sequences, for instance generalized periodic sequences (MacGregor [AMM 87 90 80]), k-automatic sequences (Cobham [MST 3 186 69; 6 164 72]), k-regular sequences (Allouche and Shallit [TCS 98 163 92]), differentiably finite sequences (Stanley [EJC 1 175 80]), constructibly differentiably finite sequences (Bergeron and Reutenauer [EJC 11 501 90]), etc., which could be used as a basis for a more rigorous classification. We should also mention the recent studies of integer sequences that have been made by Lisoněk [Liso93], Sattler [Satt94] and Théorêt [Theo94], [Theo95].

• There are many other features that could be added to the table, such as:

- Maple, Macsyma, Mathematica, Pari, etc. procedures to generate as many terms of the sequence as desired (if available), or
- a complete list of all known terms (if it is difficult to generate);
- generating functions or recurrences in every case for which they are known;
- a description of the asymptotic behavior of the sequence, and other interesting mathematical properties;
- full details of the source for each sequence (author, title, etc.), or even,
- the full text of the article or an extract from the book where the sequence appeared.

Finally, what about a table of arrays? Much remains to be done!

3.5 Acknowledgments

We thank the more than 400 correspondents who have contributed sequences to this book over the past twenty-five years. It would not be appropriate to list all their names here, but without their help, the book would not be as complete as it is.

We are especially grateful to Mira Bernstein, John Conway, Susanna Cuyler, Martin Gardner, Richard Guy (for a correspondence that spans more than 25 years), Colin Mallows, Robert Robinson, Jeffrey Shallit, and Robert Wilson (who has been our most prolific contributor of new sequences) for their assistance, as well as the late Victor Meally, John Riordan and Herman Robinson. Friends in the Unix room at Bell Labs, especially Andrew Hume and Brian Kernighan, have helped in innumerable ways. The `gfun` package of Bruno Salvy and Paul Zimmermann (see Section 2.87) has been of great help. Sue Pope typed the introductory chapters and produced LaTeX versions of many of the figures. In the summer of 1980 Bob Hinman and Theodore Peck helped in converting the sequences from the punched card format of the 1973 book.

The staff of the Bell Labs library, especially Dick Matula, have been very helpful. This great library is one of the few in the world where one can find comprehensive collections in mathematics, engineering, physics and chemistry under one roof.

Above all, N. J. A. S. wishes to thank the Mathematical Sciences Research Center at AT&T Bell Labs for its continuing support over the years. Thank you, Robert Calderbank, Mike Garey, Ron Graham, Andrew Odlyzko and Henry Pollak.

THE TABLE OF SEQUENCES

SEQUENCES OF 0's AND 1's

M0000 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
The zero sequence. [0,1; A0004]

M0001 0, 1, 0
A simple periodic sequence. [0,1; A0035]

M0002 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
The characteristic function of 0: $a(n) = 0^n$. [0,1; A0007]

M0003 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
The simplest sequence of positive numbers: the all 1's sequence. [0,1; A0012]

SEQUENCES BEGINNING . . ., 2, 0, . . .

M0004 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
The first sequence in the main table. [0,1; A0038]

M0005 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6
Theta series of diamond lattice with respect to mid-point of edge. Ref JMP 28 1653 87. [0,4; A5926]