Optimal Strategies to Detect Randomly Walking Cat

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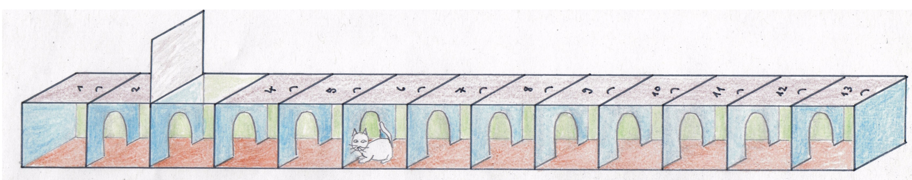
1 Introduction

In all problems studied here, a cat is hiding in one of n boxes and a single player tries to catch the cat. In one step, first the player opens a box, if the cat is found, the game is won, if not the cat will move randomly one box to the left or one box to the right. In case where the left and right boxes have exits, the cat will escape from that box with a probability of 50 %. In case where the left and right boxes do not have exits, the cat will move with 100 % probability from box 1 to 2 or from box n to $n-1$, respectively. The cat is initially assigned a random box.



Strategies are denoted as $s_1 s_2 \ldots s_k \overline{t_1 t_2 \ldots t_\ell}$, which means the boxes s_1 to s_k are opened in the steps from 1 to k and then the boxes t_1 to t_ℓ are opened repeatedly in this sequence until the cat is found or, in case where the outer boxes have escape doors, the cat has escaped.

Since the problems are symmetric, all strategies have a mirror strategy. Strategy $s_1 s_2 \ldots s_k$ will achieve the same results as strategy $n-s_1, n-s_2, \ldots n-s_k$. In this paper, the strategies where the first box opened is on the left side ($s_1 \leq \frac{n}{2}$) are presented. (Only for $n=3$ the middle box is opened first, in case of a larger odd n it is not advisable to start with opening the middle box.)

2 Walking grid with no exits

2.1 Strategy to find cat in a minimum number of steps

Sequentially opening the boxes 2, 3, 4, ..., $n-1$, $n-1$, ... 4, 3, 2 guarantees that the cat is found latest after step $2n-4$ ($n > 2$).

On average with this strategy it takes approximately $n - 1.5$ steps to find the cat: for n odd, the probability to find the cat in the first step is $\frac{1}{n}$, in the next steps up to step $n-2$ the probability is $\frac{1}{2n}$, at step $n-1$ the probability is about $\frac{n+1}{n(n-1)}$ and in the final steps the probability is about $\frac{n+1}{2n(n-1)}$. Therefore, the expectation value E is

$$E(n) = \frac{1}{n}\left(1 + \frac{1}{2}\sum_{i=2}^{n-2} i + n + 1 + \frac{1}{2}\frac{n+1}{n-1}\sum_{i=2}^{n-2}(i+n-2)\right).$$

This yields

$$E(n) = \frac{1}{n}\frac{2n^3 - 5n^2 + 3n + 2}{2(n-1)} \approx n - 1.5$$

For n even, it takes a bit longer, but eventually $E(n)$ is also approaching $n-1.5$ ($E(1000) = 998.47$ vs $E(1001) = 999.495$).

If you open the boxes randomly, even if the search could go on forever, the average number of steps to find the cat is only 1.5 larger:

$$E_{random}(n) = \frac{1}{n}\sum_{i=1}^{\infty}(i \cdot (1 - \frac{1}{n})^{i-1}) = n$$

2.2 Strategy to find cat in fastest way

The strategy presented in Section 2.1 guarantees to find the cat in no more than $2n-4$ steps, but it is not the fastest strategy if $n > 4$. A semi-automated algorithm was written, that tries all possible strategies up to a length of *maxdepth* steps, but prunes the branches where a given threshold for the average duration of the search is surpassed. Whenever a new minimum is found, the threshold is reduced.

The proof that a strategy is optimal is also performed with a search tree algorithm. For each step, it is tested whether opening a different box allows to reduce the expected duration of the game. If all alternatives lead to an increase of the game duration, then it is proved that for this step the optimum was found. Since the repetitive pattern of the strategies eventually also lead to a repetitive pattern of the cat residence probability distribution, it is only necessary to prove the optimality up to the step where the repetition starts.

The optimal strategies that minimize the duration of the search are presented in Table 1 for the number of boxes n up to 8.

n	**Strategy**	**Average duration**	**Duration of strategy 2.1**
2	11	3/2	
3	22	5/3	5/3
4	2332	39/16 (2.4375)	39/16 (2.4375)
5	$\overline{2442}$	44/15 ($2.9\overline{3}$)	71/20 (3.55)
6	$2552335\overline{5522}$	34165/9984 (3.4220)	279/64 (4.3594)
7	$263265432\overline{6325}$	7373/1792 (4.1144)	9897/1792 (5.5229)
8		4.7496	25963/4096 (6.339)

Table 1: Optimal strategy (or one of the possible optimal strategies) to catch as fast as possible the cat performing a random walk through n boxes, the average game duration of this strategy and the average game duration of the strategy 234...n-1n-1...432 where the cat is found after $2n-4$ steps at most.

The cases $n = 2$ and 3 are trivial. For $n = 4$ also strategy 22332 gives the same expected duration of 39/16.

But for $n > 4$ alternative strategies are found where the cat is found faster. For $n = 5$, strategy $\overline{2442}$ produces a relative cat residence probability of $[0.3, 0.1, 0.3, 0.3, 0]$ in steps 2, 6, 10, ..., hence the repetition cycle has a length of 4 and the proof of optimality only needs to be made for steps 1 to 5. The game situations in step 2 and 6 are identical.

For $n = 6$, strategy $2552335\overline{5522}$ produces a relative cat residence probability of $\frac{1}{66}[0, 20, 13, 15, 13, 5]$ in steps 9, 13, 17, ... The optimality of the strategy was proved for the first 12 steps which is sufficient since thereafter the game is repeating.

For $n = 7$, strategy $263265432\overline{6325}$ produces a relative cat residence probability of $[0, 0.25, 0, 0.25, 0, 0.5, 0]$ in steps 13, 17, 21... But besides its mirror strategy also strategy $225665432\overline{2563}$ produces the same results. In steps 13, 17, 21, ... the relative cat residence probability is just mirrored: $[0, 0.5, 0, 0.25, 0, 0.25, 0]$.

The game duration is reduced by a factor of 1.210 ($n = 5$), 1.274 ($n = 6$), 1.342 ($n = 7$) and 1.334 ($n = 8$) if the faster strategy is applied. An open question is whether this factor will stabilize towards a value around 1.3 or drop maybe even towards 1 when n tends to infinity.

3 Walking grid with exits on left and right side

If the cat is in the left or right box it will leave the grid with a probability of 50 %. This is called an escape. In order to find the strategy that minimizes the probability of escape, a search algorithm was written, that tries all possible strategies up to a length of *maxdepth* steps, but prunes the branches where "too many cats" have already escaped. For each strategy there is a mirror strategy that yields exactly the same result and therefore we limit the opening of the first box to the left half including the central box in case of n being odd. In all cases up to $n = 7$ one or more candidate strategies were found and an obvious repetition pattern identified.

Depending on the strategy, the cat residence distribution function starts oscillating after a while with cycles of 4 ($n = 6$), 6 ($n = 5$), or 8 ($n = 4$). For $n = 7$ it is not an exact repetition but it asymptotically approaches a cycle of length 4. Therefore, it is possible to calculate the exact percentage of cats that can escape. The optimal strategies and the escape rates are:

n	**Strategy**	**Escape rate**	**Average duration**
3	22	1/3	1.33333
4	$\overline{14414114}$	1105/3968 (0.27848)	2.83543
5	$144\overline{141}$	9643/37120 (0.25978)	2.79168
6	$15261\overline{2552}$	305/1248 (0.24439)	3.61619
7	$1661\overline{2266}$	183/784 (0.23342)	4.54974
8	$177122477\overline{s_{22}}$	0.22331	5.34690

Table 2: Optimal strategy (or one of the possible optimal strategies) to minimize the escape rate of a cat performing a random walk through n boxes, the resulting escape rate and the average game duration (in case of 8 boxes only the first 57 steps of the strategy were proved, the 22 digits of strategy s_{22} are given in the text below).

To prove that the candidate strategy is the optimum, each decision step of the strategy is modified and it is tested if opening a different box at this step will allow to reduce the escape rate. Since we have a tentative minimum given in Table 1, any branch of the search tree can be pruned as soon as the escape rate exceeds the presumed minimum. Therefore, for $n < 7$, the proof of the optimal strategy was relatively quick.

For $n = 3$, opening twice box 2 yields an escape rate of 1/3, in two of three cases the cat will be caught. But also opening box 1 continuously gives the same escape rate. Or opening box 1 for some time and then opening twice box 2.

For $n = 4$, strategy $\overline{14414114}$ proves to be the only optimum (except for its mirror strategy $\overline{41141441}$). The cat residence distribution function after step 3 is: $\frac{1}{32}[3, 2, 3, 1]$ and after step 11 it is $\frac{1}{1024}[3, 2, 3, 1]$. The function has decreased by a factor of 32 in these 8 steps and since the strategy has also a cycle of length 8, this pattern will repeat for ever. Hence we have only to test the first 11 decision steps. And for all 11 steps it turned out that any modification lead to an increase in the escape rate latest at step 16.

For $n = 5$, there are 4 equivalent strategies (plus their 4 mirror strategies) for the first 9 steps: $14s_1s_2141$ where $s_{1,2} \in \{25, 41\}$ (the residence probability after step 9 always is: $\frac{9}{2560}[1, 1, 2, 1, 1]$). The proof is straightforward: all strategies which are different lead to an increase in the escape rate latest at step 14. Strategies like 1445 and 1421 have an escape rate larger than 0.26 already after 4 steps.

After step 9 there are the two mirror strategies 141 and 525 that both return a symmetric residence probability of the form $k_1 \cdot [3, 2, 6, 2, 3]$ and applied a second time, of the form $k_2 \cdot [1, 1, 2, 1, 1]$. Hence we have a repeating cycle for the residence probability of length 6. Checking all possible different strategies up to step 14 where the repetition cycle starts, revealed that all lead to higher escape rates latest after 19 steps. In summary, the optimal strategies are: $14s_1s_2141s_3s_4s_5...$ where $s_i \in \{141, 525\}$ for $i > 2$.

For $n = 6$, strategy $15261\overline{2552}$ proves to be the only optimum (except for its mirror strategy). The cat residence distribution function after step 9 is: $\frac{1}{768}[0, 12, 5, 12, 5, 4]$ and after step 13 it is $\frac{1}{4096}[0, 12, 5, 12, 5, 4]$. The function has decreased by a factor of $\frac{16}{3}$ in these 4 steps and this pattern will repeat for ever. Hence we have only to test the first 13 decision steps. And for all 13 steps it turned out that any modification leads to an increase in the escape rate latest at step 29.

For $n = 7$, strategies $1661\overline{2266}$ and $1627\overline{6226}$ are two equivalent strategies. The respective residence probability distribution reveal a one-to-one mapping between the two strategies and therefore we confine our analysis to the first strategy. Since the opening of the boxes repeat with a cycle of length 4 we can compute the evolution of the residence probability with the following transition matrix T:

$$T = \frac{1}{16}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 5 & 0 & 2 & 0 & 1 \\ 1 & 0 & 4 & 0 & 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

where T_{ij} defines which fraction of box i is moved to box j in the 4 steps when boxes 2, 2, 6 and 6 are opened. "Box 8" counts the fraction of cats that escape. (Note, since all contributions from box 2 are removed when this box is opened in the first step of the repetition cycle, row 2 is empty and could be removed from the calculations. But since the calculations are performed almost instantaneously, we keep it included for better readability.)

The matrix T can be diagonalized:

$$T = PDP^{-1}$$

where D is the diagonal matrix with the eigenvalues $[0\ e_1\ e_2\ e_2\ e_1\ 0\ 0\ 1]$ of the matrix T. $e_1 = \frac{3-2\sqrt{2}}{16} \approx 0.010723$ and $e_2 = \frac{3+2\sqrt{2}}{16} \approx 0.364277$.

The initial residence distribution p_1 after opening the boxes 1, 6, 6, 1 becomes:

$$p_1 = \frac{1}{7 \cdot 32}[8, 14, 18, 22, 14, 8, 4, 36]$$

The escape rate can be calculated as the last component of the vector p_∞:

$$p_\infty = \lim_{n\to\infty} T^n p_1 = \lim_{n\to\infty} PD^nP^{-1}p_1$$

Since all eigenvalues except the last one are smaller than 1, $\lim_{n\to\infty} D^n$ is a matrix with zeros except for $D^\infty_{88} = 1$. Hence the escape rate becomes

$$(P^{-1})_{88} \sum_{i=1}^{8} P_{i8} \cdot p_{1i} = \frac{183}{784} \approx 0.23341837.$$

Any modification of the conjectured optimal strategies in the first 50 steps leads to an increase in the escape rate latest 11 steps later. A sketch of a proof that this holds for all steps beyond 50 is provided now.

The relative residence probability (relative with respect to the remaining fraction of cats still in the boxes) converges towards this repetitive pattern:

Step t	**Box 1**	**Box 2**	**Box 3**	**Box 4**	**Box 5**	**Box 6**	**Box 7**
$4i$	k_1	k_2	k_3	k_4	k_5	k_6	0
$4i+1$	0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
$4i+2$	0	k_6	k_5	k_4	k_3	k_2	k_1
$4i+3$	ℓ_6	ℓ_5	ℓ_4	ℓ_3	ℓ_2	ℓ_1	0

Table 3: Asymptotic relative residence probability for strategy $1661\overline{2266}$.

The decrease factor of the absolute residence probability between 4 steps approaches $48 - 32\sqrt{2} = e_2^{-1} \approx 2.745166$. The vector k in Table 3 is

$$k = \frac{1}{1447}[893 - 563\sqrt{2}, -177 + 319\sqrt{2}, 660 - 233\sqrt{2}, 147 + 152\sqrt{2}, -233 + 330\sqrt{2}, 157 - 5\sqrt{2}]$$

or numerically:

$$k = [0.06690, 0.18945, 0.22840, 0.25015, 0.16150, 0.10361].$$

The vector ℓ in Table 3 is

$$\ell = \frac{1}{255}[163 - 81\sqrt{2}, 17 + 17\sqrt{2}, 47 + 12\sqrt{2}, 34 + 17\sqrt{2}, -23 + 35\sqrt{2}, 17]$$

or numerically:

$$\ell = [0.18999, 0.16095, 0.25086, 0.22761, 0.10391, 0.06667].$$

Taking the asymptotic relative residence probability and applying the strategy $1661\overline{2266}$ gives an escape rate of $\frac{1}{1785}(103 + 116\sqrt{2}) \approx 0.14961$. Trying out all possible other strategies gives a minimum escape rate of 0.15014 in the following 12 steps if we start at step $4i$ (or $4i + 2$ which is a symmetric case) and 0.14994 if we start at step $4i + 1$ (or $4i + 3$).

We analyse first the case of the conjectured optimal strategy at step $4i+1$. It can be observed that the relative residence probabilities in boxes 3 to 7 approach the values ℓ_2 to ℓ_6 from the upper side when i tends to infinity and that the relative residence probability in box 2 is larger than $\ell_1 \cdot (1-\epsilon)$ with $\epsilon = 42 \cdot (1+\sqrt{2})^{-t}$.

The minimum escape rate of the asymptotic residence distribution of any modified strategy was 0.14994 during the next 12 steps. If boxes 3 to 7 have at step $4i + 1$ larger probabilities than the asymptotic distribution (even only marginally) and box 2 has a probability larger than $\ell_1 \cdot (1 - \epsilon)$, then the escape probability for any modified strategy during the next 12 steps is larger than $0.14994 - \ell_1 \cdot \epsilon$. The inequality

$$0.14994 - \ell_1 \cdot 42 \cdot (1 + \sqrt{2})^{-t} > 0.14961$$

holds for all $t > 30$ and therefore all modified strategies have higher escape rates than the optimal strategy $1661\overline{2266}$. Hence this strategy is optimal at all steps $4i + 1$.

The same reasoning must be applied for the three other cases $4i$, $4i + 2$ and $4i + 3$, which is not shown here.

For $n = 8$ no short repetitive strategy pattern was found. The optimal strategy for the first 57 steps is 177122477234718723776223681876123471872377 622368187612347. The strategy $s_{22} = \overline{2347187237762236818761}$ may have to be applied repetitively starting at step 10, but this was not proved (the escape rate for this strategy is 0.22330599). In step 58 either box 1 or 7 should be opened.

If the strategy $s_5 = \overline{72347}$ is applied from step 58 onwards, the relative residence probability distribution starts repeating from step 62 onwards with a cycle length of 10. The escape rate for this strategy is 0.22330600. The strategy s_5 transforms any arbitrary cat residence distribution into a distribution of the form $[5q_1, 0, 15q_1, q_2, 20q_1, q_2, 14q_1, 0]$ after 10 steps. This distribution repeats after another 10 steps, only reduced by a factor of $\frac{1024}{75}$, hence a cycle length of 10 is established.

Both identified strategies provide an upper bound for the escape rate of 0.2233060. A lower bound is 0.2233059 for which no strategy can be found. It should be noted that the average game duration is only 5.35 steps and the probability that the game is not yet finished in the first 57 steps is only 8.34×10^{-6}.