ILLUSTRATION OF FACE-MAGIC CUBES

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Abstract. Face-magic cubes of order 2 are arrangements of 8 distinct positive integer numbers at the vertices of a cube where the sum of the 4 elements of each of the 6 sides is the same. Order 2 means that each of the 12 edges hosts two integers. We print these cubes up to magic sum 29 for the special case where the 8 integers are distinct, start at 1, and are reduced by the point group symmetries of the cube.

1. Definition

A face-magic cube of order 2 has 8 positive integer numbers, one at each vertex of the cube, such that the 6 sums over the 4 vertices are the same for each of the 6 faces; that sum is the magic number n.

Remark 1. *One can rephrase this for non-negative integers, which merely shifts the results by 4 in the magic sums: adding 1 to each of the vertices increases the magic sum by 4.*

It seems that one can construct 1, 4, 12, 28, 57... of these face-magic cubes for $n = 4, 5, 6, \ldots$ [1, A203286].

If we require that the 8 integer numbers are distinct, we get 144, 96, 288, 288, 912, 720... magic cubes for $n = 18, 19, 20, \ldots$.

If we require that cubes that are siblings created by any of the 48 point operations of the full octahedral group (24 pure rotations plus mirror images through space of face diagonal planes) are counted only once, we apparently get 1, 1, 3, 4, 8, 10,... magic cubes for $n = 4, 5, 6 \ldots$[1, A115264]

If we require both (awareness of symmetry and distinctness) we obtain 3, 2, 6, 6, 19, 15, 27, 34, 57, 55, 84, 92, 138, 147, 192, 210, 288... magic cubes for $n = 18, 19, 20 \ldots$.

Conjecture 1. *These symmetry-aware face-magic cubes or order 2 obey the C-finite recurrence*

(1)
$$\begin{aligned} b(n) = &-b(n-1)-b(n-2)+b(n-4)+2b(n-5)+3b(n-6)+3b(n-7)+3b(n-8)+b(n-9)\\ &-b(n-10)-4b(n-11)-5b(n-12)-6b(n-13)-5b(n-14)-3b(n-15)+3b(n-17)+5b(n-18)\\ &+6b(n-19)+5b(n-20)+4b(n-21)+b(n-22)-b(n-23)-3b(n-24)-3b(n-25)-3b(n-26)\\ &-2b(n-27)-b(n-28)+b(n-30)+b(n-31)+b(n-32). \end{aligned}$$

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equivalent to the rational ordinary generating function

$$(2)\quad x^{18}(14x^{14}+19x^{13}+29x^{12}+32x^5+28x^4+14x^3+11x^2+5x+31x^{11} \\ +47x^{10}+48x^9+54x^8+43x^7+42x^6+3) \\ /[(x^2-x+1)(x^4+x^3+x^2+x+1)(x^4+1)(x^6+x^5+x^4+x^3+x^2+x+1) \\ (x^2+1)^2(1+x+x^2)^2(1+x)^3(1-x)^5].$$

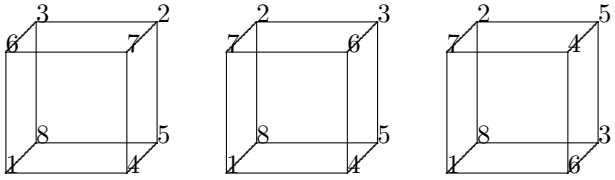
According to remark 1 these contain copies where the smallest integer is not 1, so we define another sequence which recognizes symmetry, distinctness, and in addition that the smallest of the 8 integers is 1 (which might be called primitive face-magic cubes): we have $a(n) = 3, 2, 6, 6, 16, 13, 21, 28, 38, 40, 57, 58, \ldots$ primitive magic cubes for magic sum $n = 18, 19, 20 \ldots$, where with Remark 1 $a(n) = b(n) - b(n-4)$. The conjectured ordinary generating function is derived from (2) by multipliation with $1 - x^4 = (1-x)(x+1)(1+x^2)$:

Conjecture 2. *These symmetry-aware primitive face-magic cubes or order 2 have a C-finite linear recurrence with the rational ordinary generating function*

$$(3)\quad x^{18}(14x^{14}+19x^{13}+29x^{12}+32x^5+28x^4+14x^3+11x^2+5x+31x^{11} \\ +47x^{10}+48x^9+54x^8+43x^7+42x^6+3) \\ /[(x^2-x+1)(x^4+x^3+x^2+x+1)(x^4+1)(x^6+x^5+x^4+x^3+x^2+x+1) \\ (x^2+1)(1+x+x^2)^2(1+x)^2(x-1)^4].$$

The subsequence illustrations display these $a(n)$ different solutions for $n = 18 \ldots 29$. The orientations are normalized such that reading the integers ccw on the base face and then reading the integers ccw on the top face gives the lexicographically smallest cube out of 48.

2. Magic face sum 18



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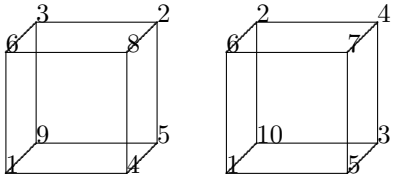

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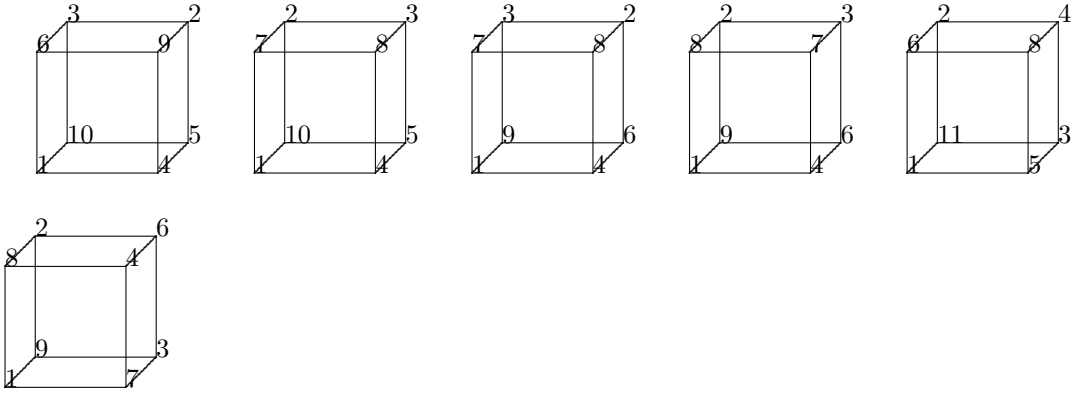

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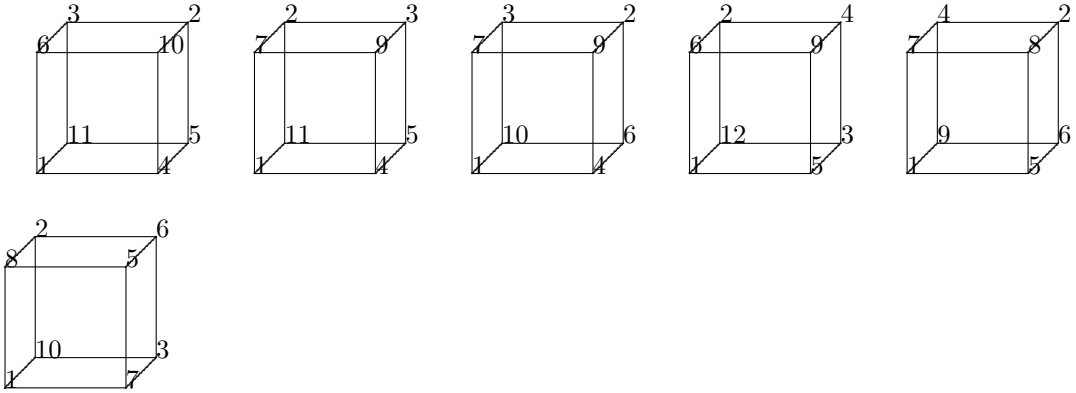

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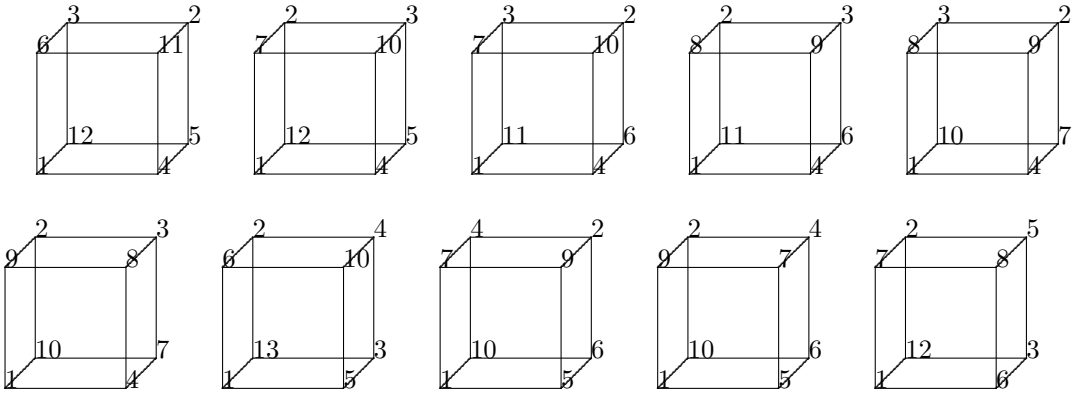

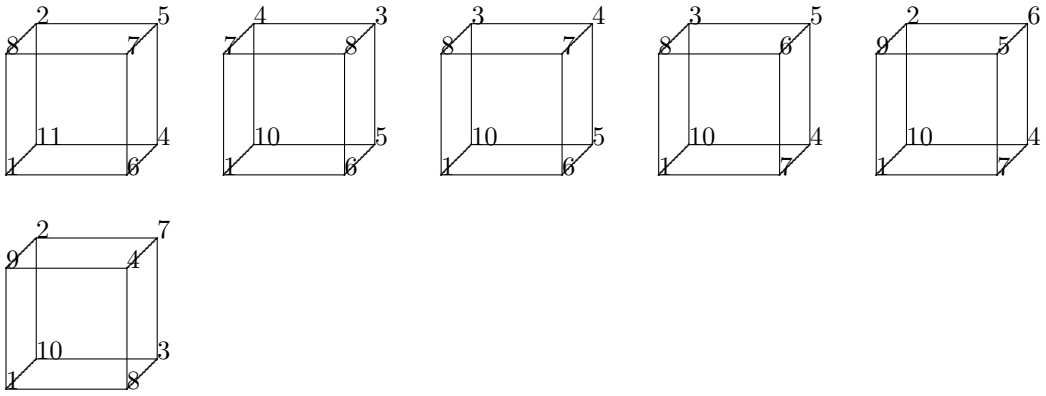

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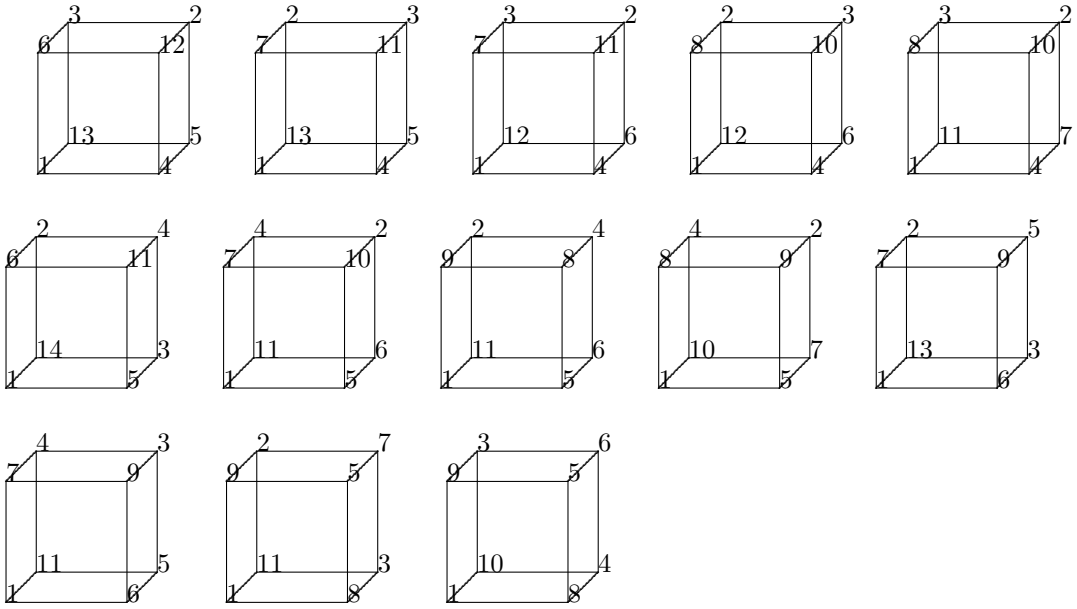

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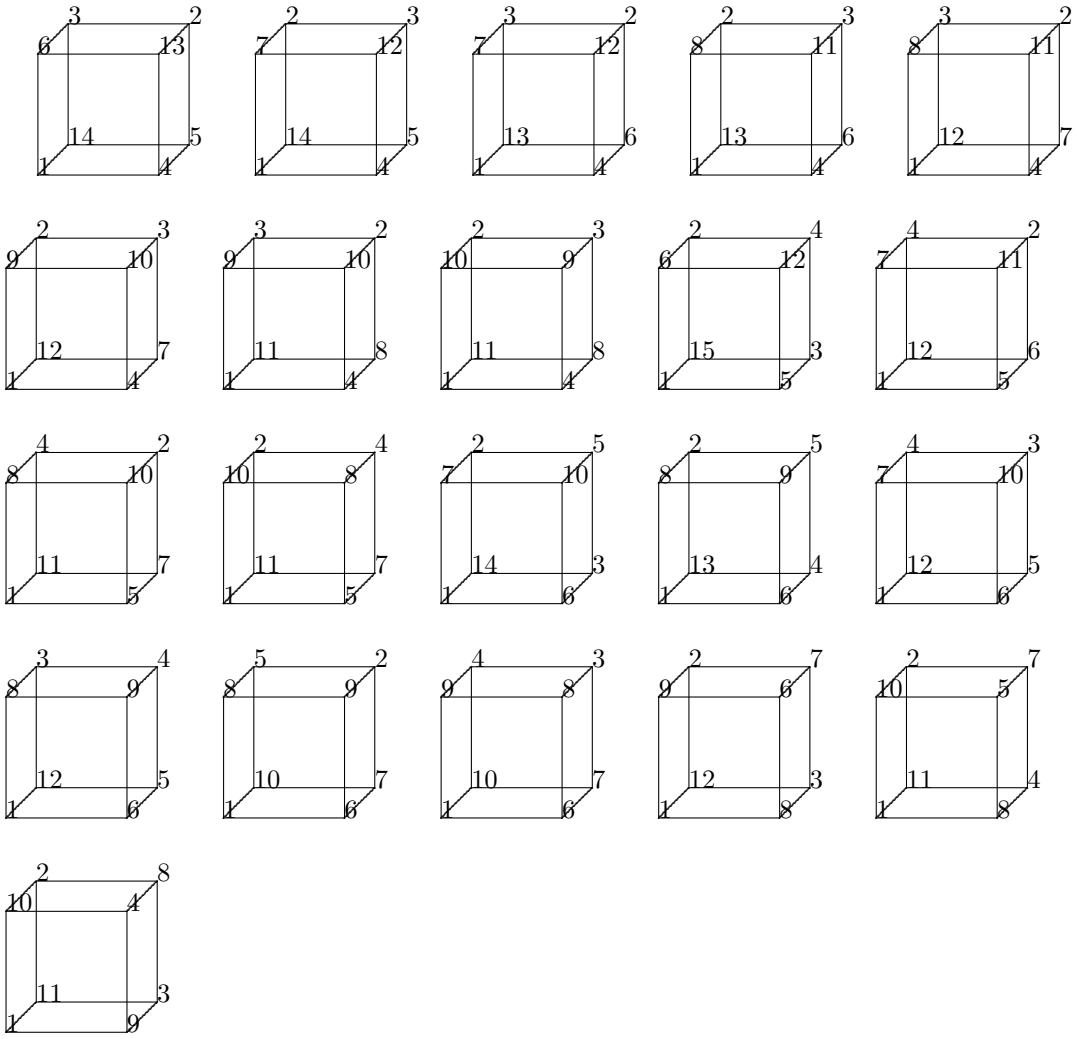

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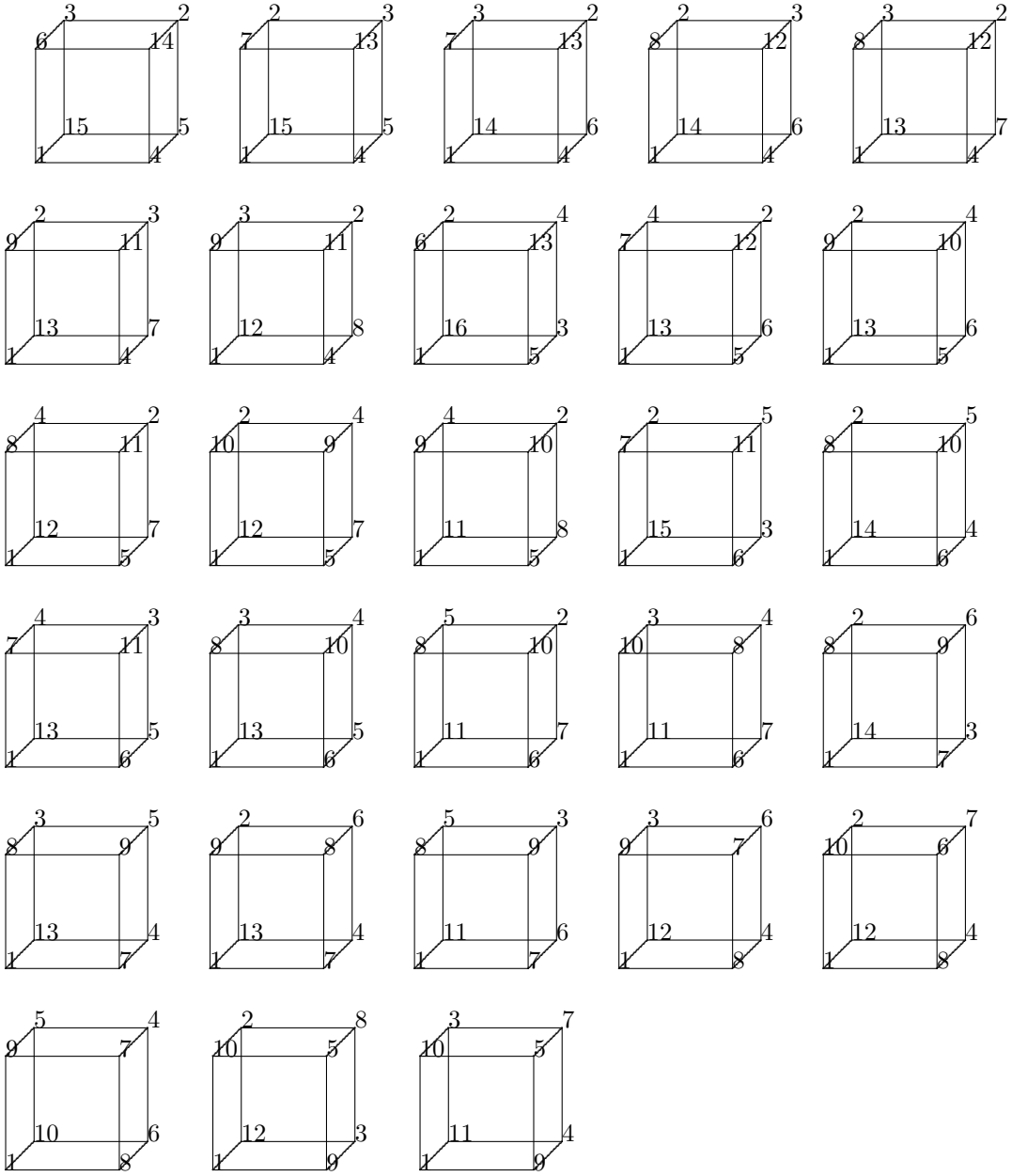

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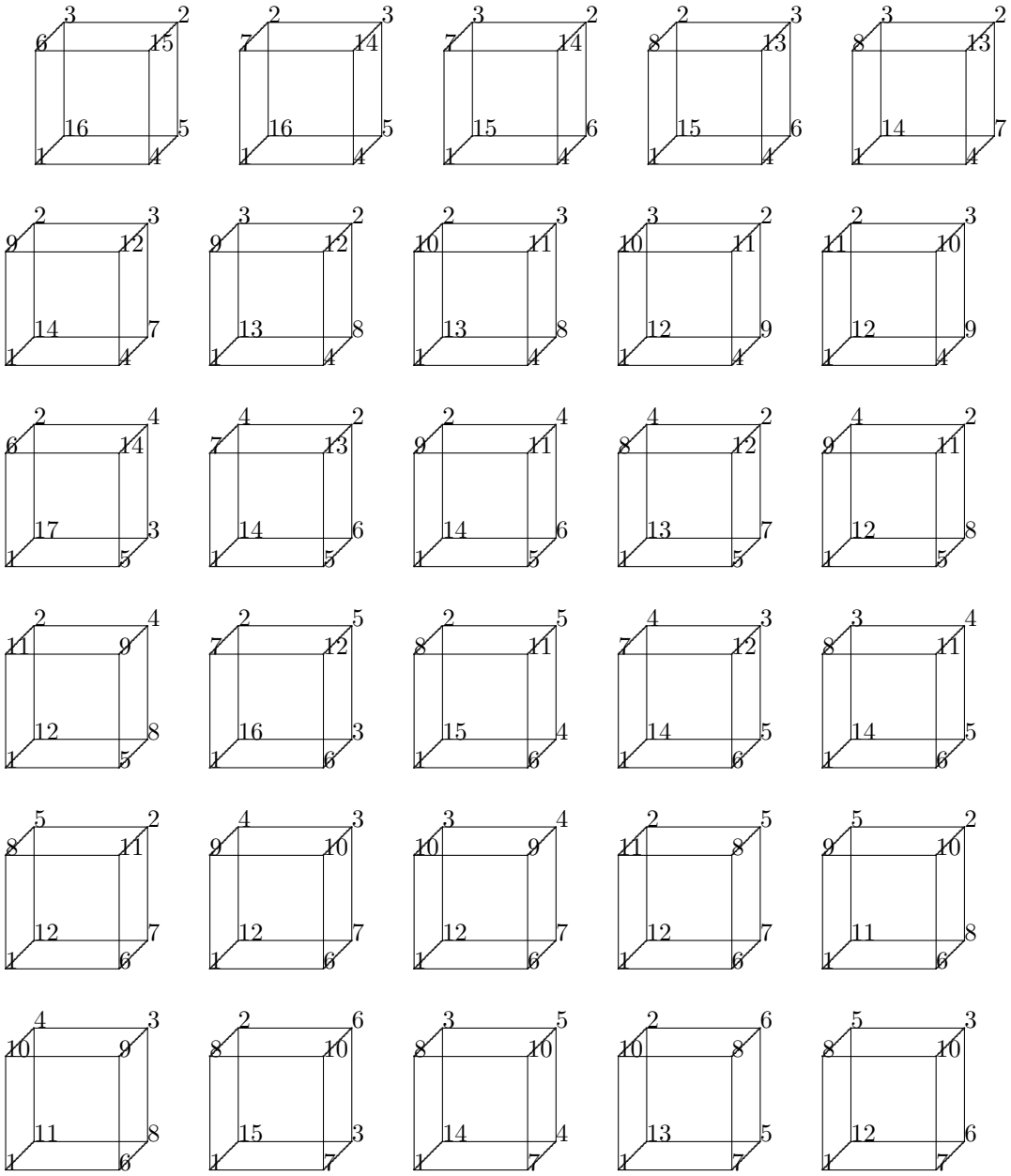
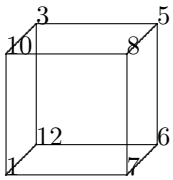
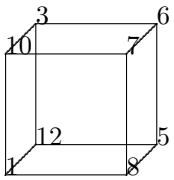
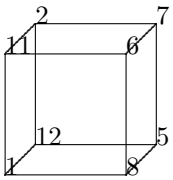
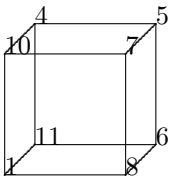
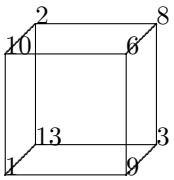


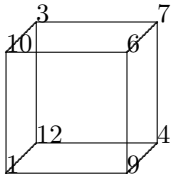


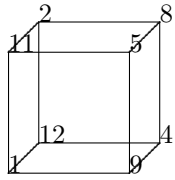


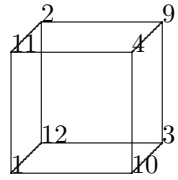




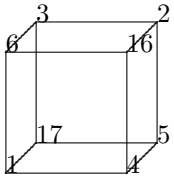


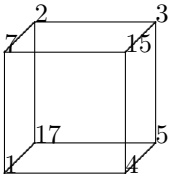


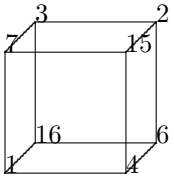


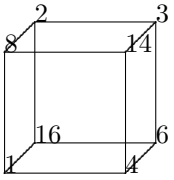


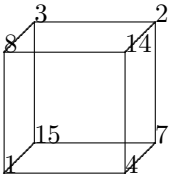
11. Magic face sum 27

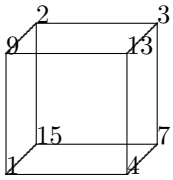


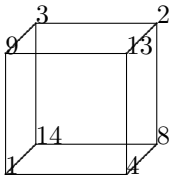


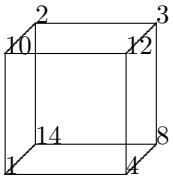


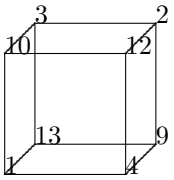


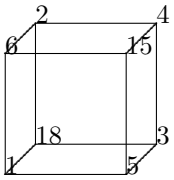


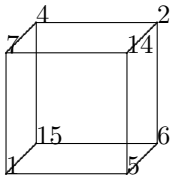


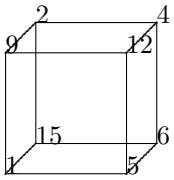


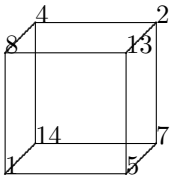


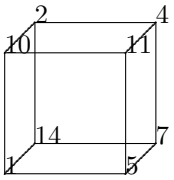


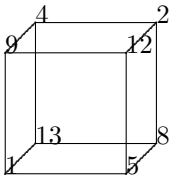


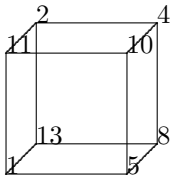


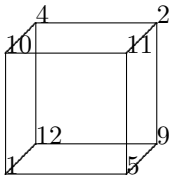


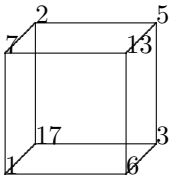


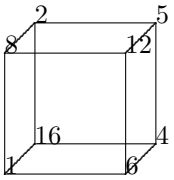


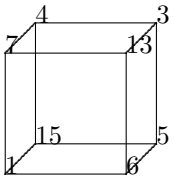


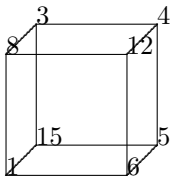


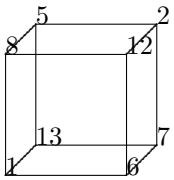


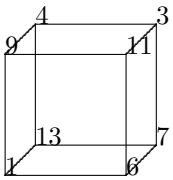


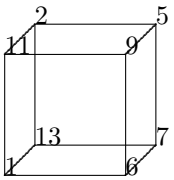


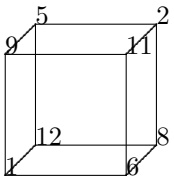


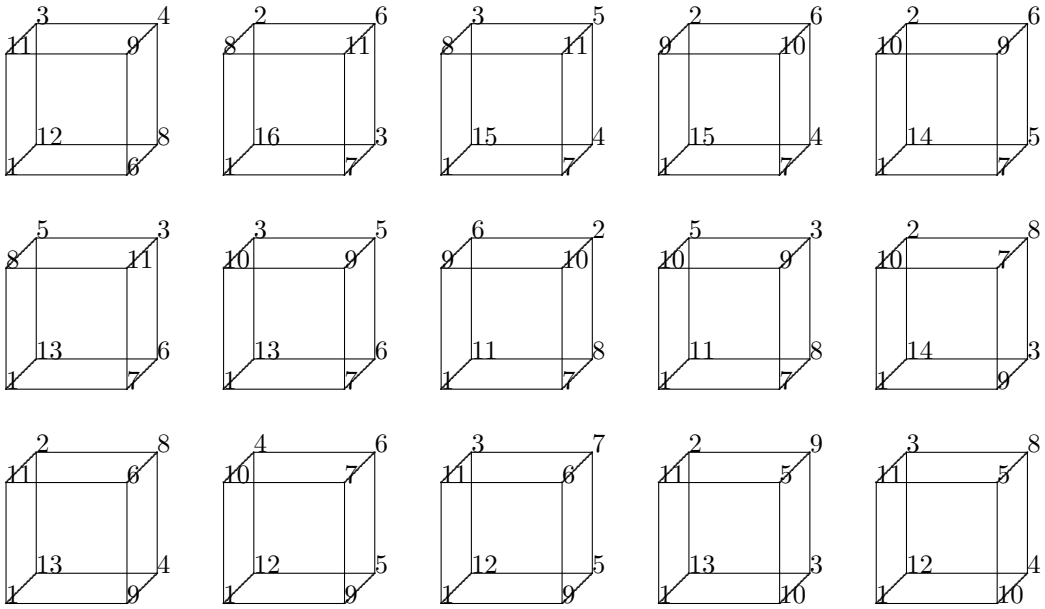




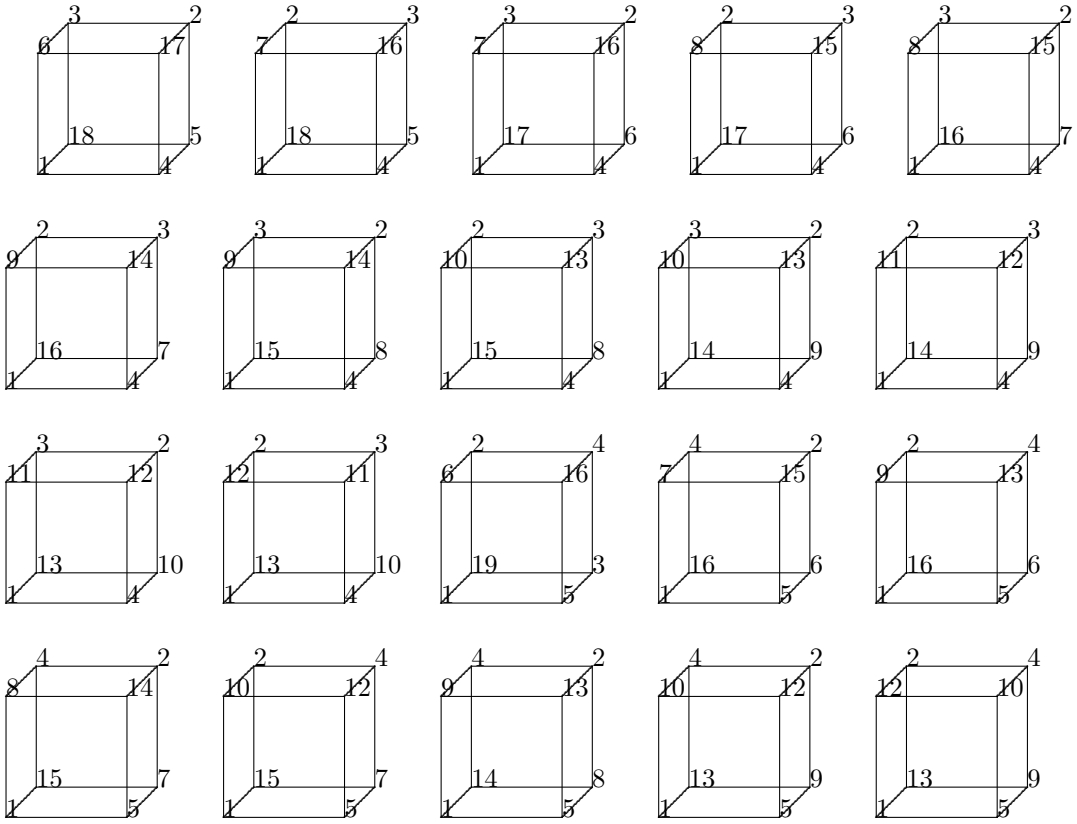


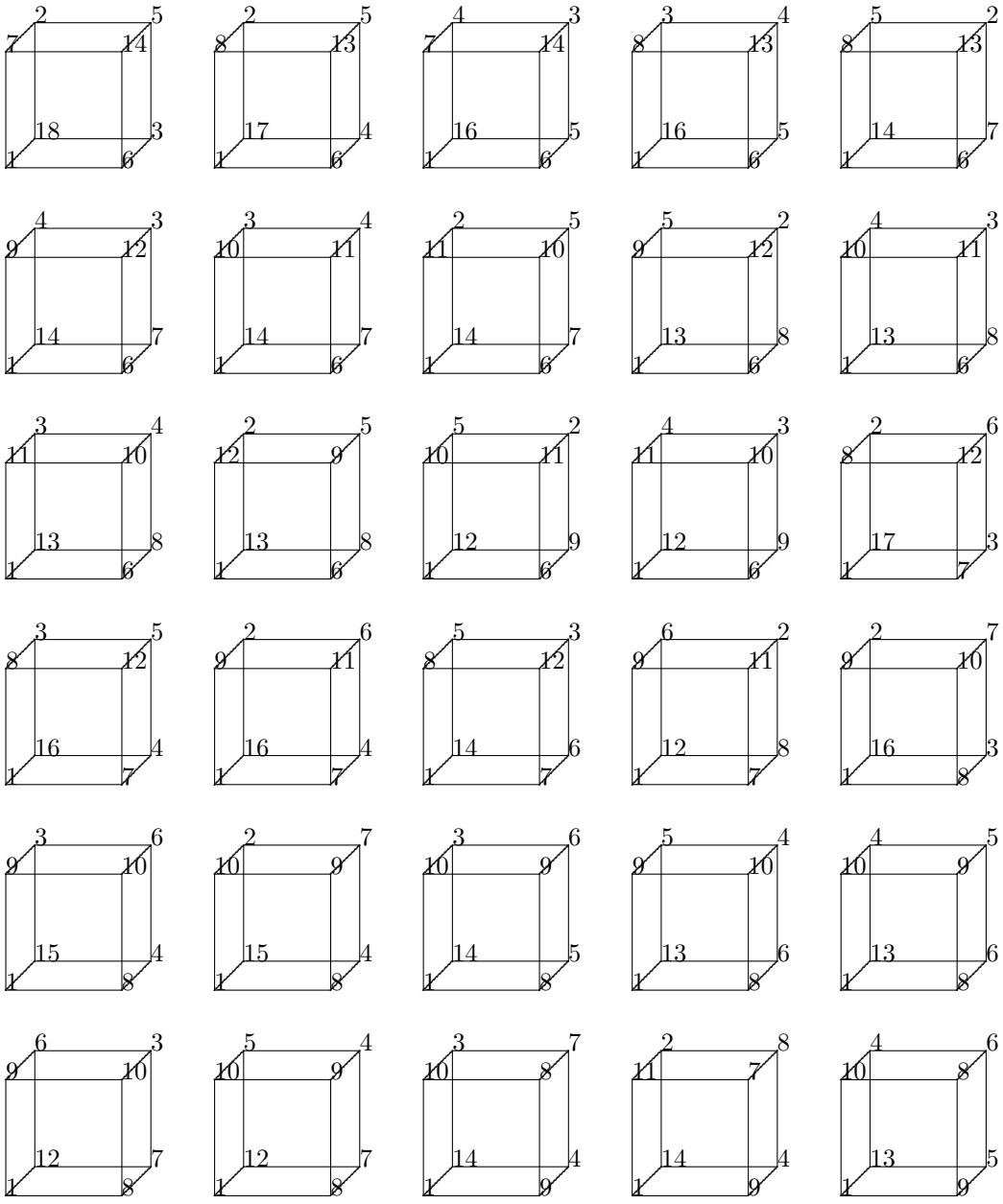






12. Magic face sum 28

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2 5 11 10 14 7 1 6
5 2 9 12 13 8 1 6
4 3 10 11 13 8 1 6
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2 6 8 12 17 3 1 7
3 5 8 12 16 4 1 7
2 6 9 11 16 4 1 7
5 3 8 12 14 6 1 7
6 2 9 11 12 8 1 7
2 7 9 10 16 3 1 8
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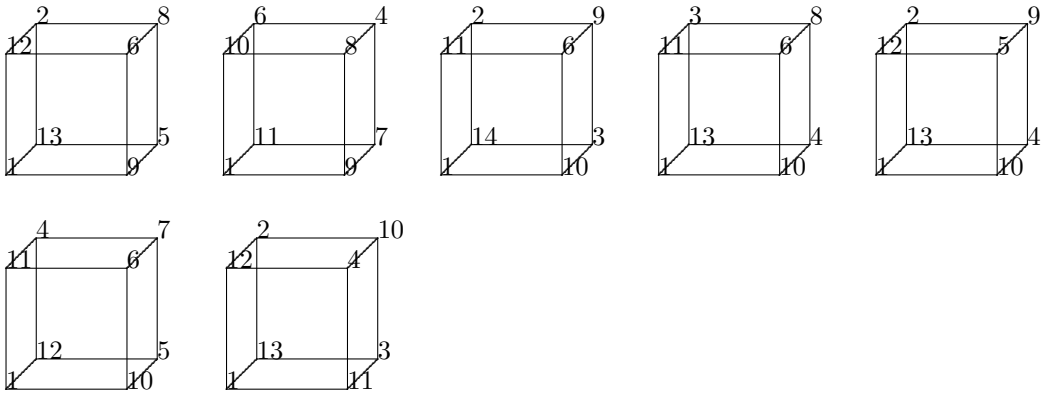
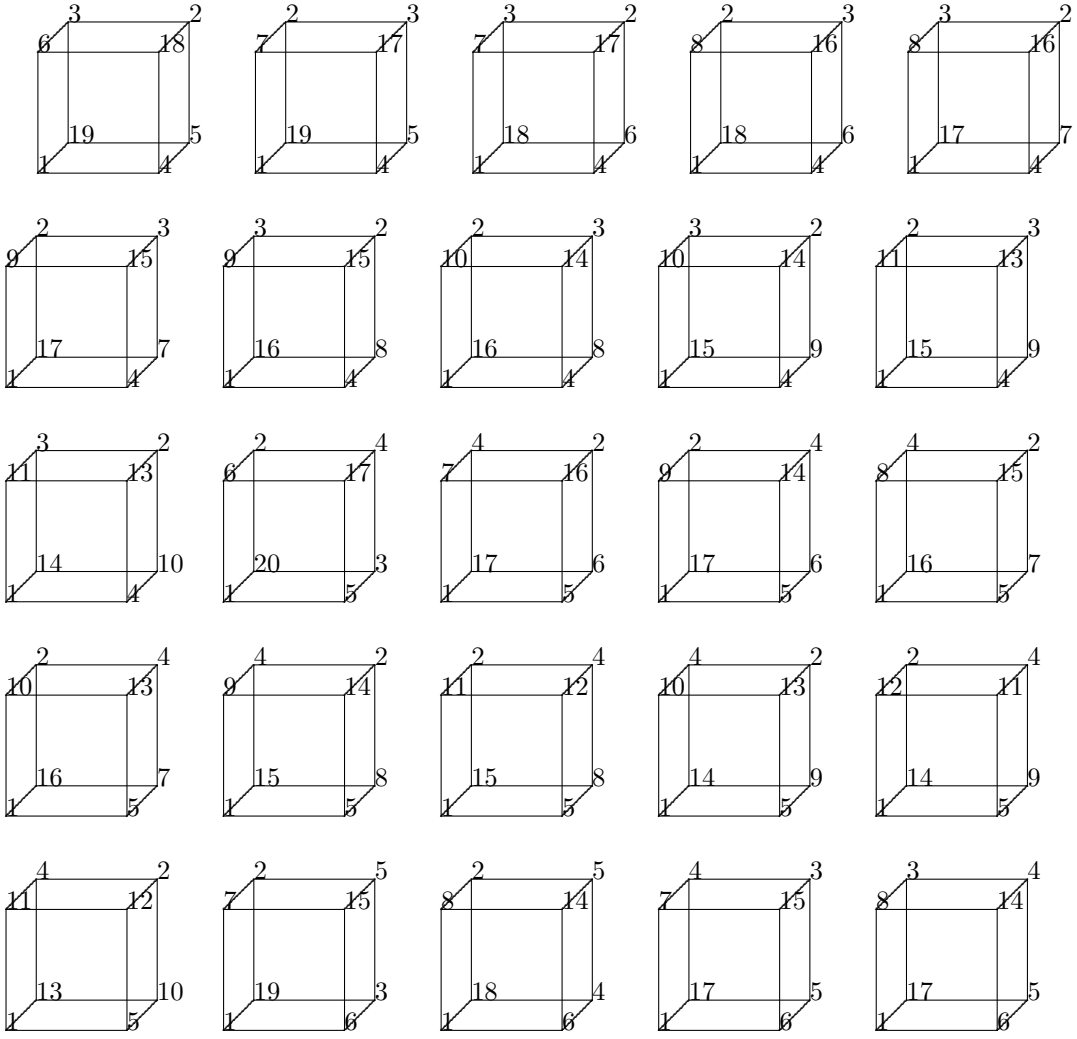
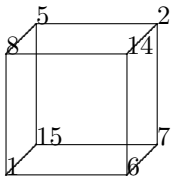


13. Magic face sum 29

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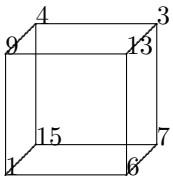
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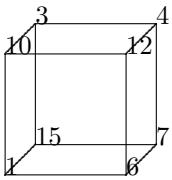
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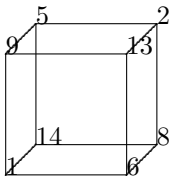
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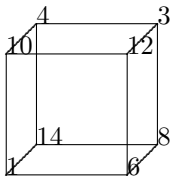
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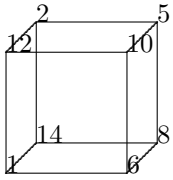
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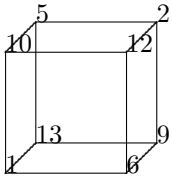
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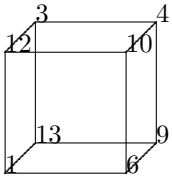
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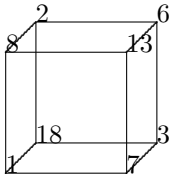
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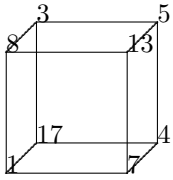
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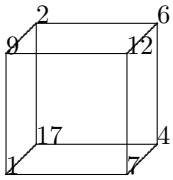
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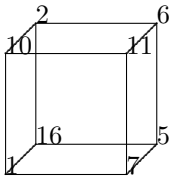
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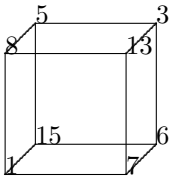
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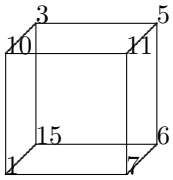
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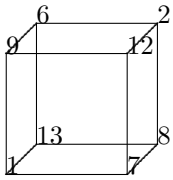
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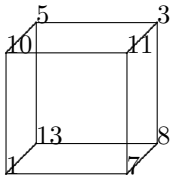
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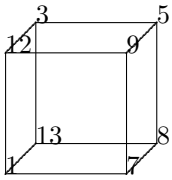
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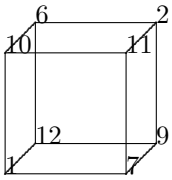
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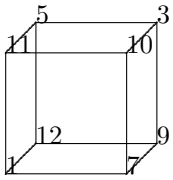
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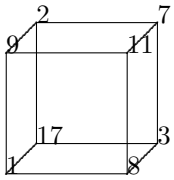
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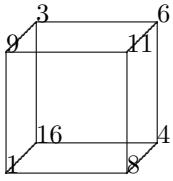
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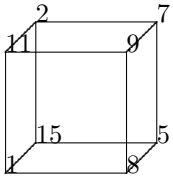
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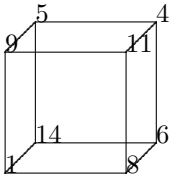
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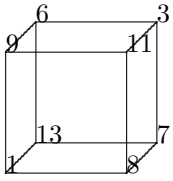
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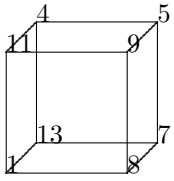
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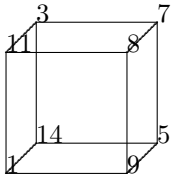
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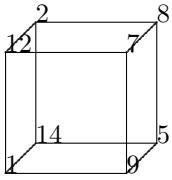
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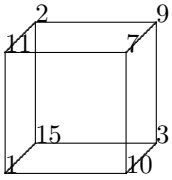
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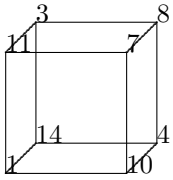
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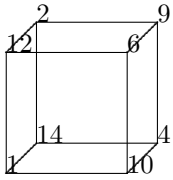
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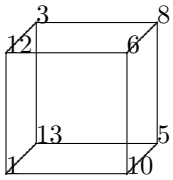
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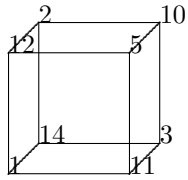
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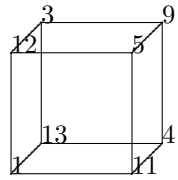
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