Notes on A380290 and A380291

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Define three infinite products $F_{\pm}(x)$ and $G(x)$ by

$$F_{\pm}(x) = \prod_{k=1}^{\infty} \left(1 \pm x^k\right)^{k^2} \text{ and } G(x) = \prod_{k=0}^{\infty} \left(1 + x^{2k+1}\right)^{(2k+1)^2}.$$

Let A be an integer. Using the coefficient extraction operator we define three sequences by

$$[x^n] \left(F_{\pm}(x)\right)^{An} \text{ and } [x^n] \left(G(x)\right)^{An}, \text{ for } n \geq 1.$$

We conjecture that each of these sequences satisfies the following supercongruences :

$$u\left(np^r\right) \equiv u\left(np^{r-1}\right) \left(\bmod\ p^{3r}\right) \ n, r \in \mathbb{N} \text{ and } p \geq 7 \text{ prime.} \qquad (1)$$

The sequences given by $[x^n]\, F_{-}(x)^{-n}$and $[x^n]\, F_{+}(x)^{n}$ have been submitted to the OEIS as A389290 and A389291.

Further, let A and B be integers and define a pair of sequences by

$$[x^n] \left(F_{+}(x)^A G(x)^B\right)^n \text{ and } [x^n] \left(F_{-}(x)^A G(x)^B\right)^n$$

We conjecture that the supercongruences (1) also hold for this pair of sequences.

More generally, let $m \neq 2$ be a positive integer and consider the three infinite products

$$F_{\pm}(m, x) = \prod_{k=1}^{\infty} \left(1 \pm x^k\right)^{k^m} \text{ and } G(m, x) = \prod_{k=0}^{\infty} \left(1 + x^{2k+1}\right)^{(2k+1)^m}.$$

As above, we can define families of sequences using the coefficient extraction operator. Calculation suggests that for these sequences the weaker congruences

$$u\left(np^r\right) \equiv u\left(np^{r-1}\right) \left(\bmod\ p^{2r}\right) \ n, r \in \mathbb{N} \text{ and } p \geq 7 \text{ prime} \qquad (2)$$

hold .