Construction of a point x with circles

1. For any representation $(y(1) = 2, y(2), .., y(m) = x)$ of x:

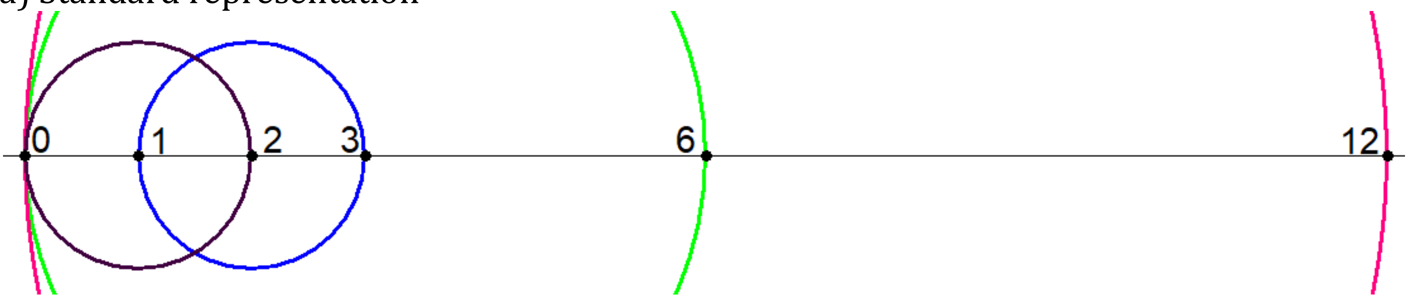
$$(1.1)\ y(j) \le 2 \cdot y(j-1), 1 < j \le m \qquad (1.2)\ 2^{j-1} < y(j) \le 2^j, 1 \le j \le m$$

2. Short description of the backward algorithm: $y(j) = \left\lfloor \frac{y(j+1)+1}{2} \right\rfloor, 1 \le j \le m-1.$

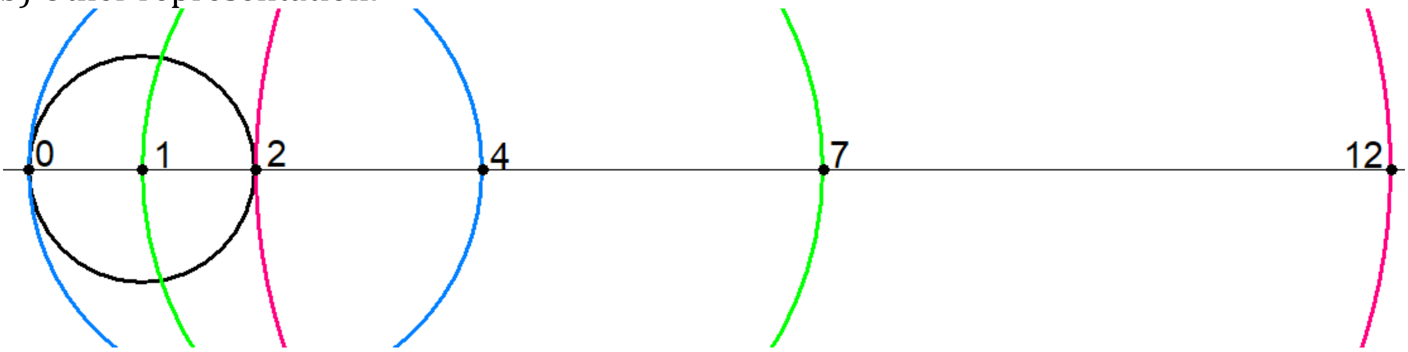
3. For even x with $2^{m-1} < x < 2^m$, at least two representations exist. Referring to the standard representation, let the last circle intersect the x-axis at 2 instead of 0. Then

$$y(m-1) = \frac{x+2}{2} \le 2^{m-1} \text{ such that (1.2) is satisfied for } j = m-1.$$

Using the backward algorithm, the representation can be completed.
See counterexample 4 for x=48 or x=12 here:
a) Standard representation



b) Other representation:



4. If the representation of x is unique, no even numbers occur, except for powers of 2.
According to the standard representation, k<m exists such that
$y(j)$ is the j-th power of 2 for $j \le m-k$ and odd for $j > m-k$.
For odd numbers, the algorithm can be read forward such that $y(j) = 2y(j-1) - 1$.
Explicitely:

$$(4.1) \qquad y(j) = \begin{Bmatrix} 2^j \ for\ j \le m-k \\ 2^j - 2^{k+j-m} + 1\ for\ j > m-k \end{Bmatrix}$$

For $j = m$: $x = y(m) = 2^m - 2^k + 1$.
This form is necessary for a unique representation of x, but not sufficient.

5. The representation of $x = 2^m - 2^k + 1$ is ambiguous for $k > \frac{m+1}{2}$.
In the standard representation, the last circle intersects the x-axis at x and 1. Instead of 1, we try the least odd number $y(m-k+1) = 2^{m-k+1} - 1$ in the standard representation.
The center of the circle then is

$$\tilde{y}(m-1) = ½\big(x + y(m-k+1)\big) = ½(2^m - 2^k + 1 + 2^{m-k+1} - 1)$$
$$= 2^{m-k}(2^{k-1} - 2^{2k-m-1} + 1)$$

Continued division by 2, m-k times, leads to $\tilde{y}(k-1) = 2^{k-1} - 2^{2k-m-1} + 1$.
This equals $y(k-1)$ for $j = k-1$ in (4.1). The current representation can be completed with $\tilde{y}(j) = y(j)$ for $j < k-1$. Note that $2k - m - 1 > 0$ requires $k > \frac{m+1}{2}$.

6. The representation of $x = 2^m - 2^k + 1$ is unique for $0 < k \le \frac{m+1}{2}$.
Assumption: The last circle intersects the x-axis at x and an odd number $\tilde{y}(j) > 1$.
Then m-1-j steps (circles) lead to $\tilde{y}(m-1) = ½(x + \tilde{y}(j))$.
With (1.1): $\tilde{y}(j) \cdot 2^{m-1-j} \ge \tilde{y}(m-1)$

$$\Rightarrow \tilde{y}(j)\big(2^{m-j} - 1\big) \ge 2^m - 2^k + 1$$
$$\Rightarrow \big(2^j - 1\big)\big(2^{m-j} - 1\big) \ge 2^m - 2^k + 1 \text{ because } \tilde{y}(j) \le 2^j - 1$$

or simplified: (6.1) $2^{m-j} + 2^j \le 2^k \le 2^{\lfloor (m+1)/2 \rfloor}$
The minimum of $2^{m-j} + 2^j$ is $2 \cdot 2^{m/2}$ if m is even and $j = m/2$
or $2^{(m+1)/2} + 2^{(m-1)/2}$ if m is odd and $j = (m \pm 1)/2$.
Both cases are contradictory to (6.1) and to the assumption.
Therefore the representation of x is unique.
Example for $x = 2^4 - 2^2 + 1 = 13$: