If $n = p^j$ where p is a prime ≥ 5, then $A379953(n) = 0$

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Suppose $n = p^j$ where p is prime, and $n^3 + k^3 \mid (nk)^3$ with $k > 0$. Then $n^3 + k^3 \mid n^6$ also. But the only divisors of n^6 are powers of p, so $n^3 + k^3$ is a power of p, i.e. $p^{3j} + k^3 = p^m$ for some $m \geq 3j$. Now $k^3 = p^m - p^{3j}$, the right side is divisible by p^{3j}, so k is divisible by p^j, say $k = rp^j$. Dividing by p^{3j}, we have

$$r^3 = p^{m-3j} - 1$$

We know $m \geq 3j$. We can't have $m = 3j$, as that would make $r = 0$. If $m = 3j+1$, then $p = r^3 + 1$. But $r^3 + 1 = (r+1)(r^2 - r + 1)$ is prime only when $r = 1$ (which corresponds to $p = 2$. For $m > 3j + 1$, Mihăilescu's theorem says the only solution of $r^3 = p^{m-3j} - 1$ in positive integers is $r = 2$. $p = 3$, $m - 3j = 2$. So if $p \geq 5$, it is impossible to have $k > 0$, and thus $A379953(n) = 0$.