Number of possibilities to tile the plane with "hat" monotiles

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November 2024

1 Introduction

In 2023 Smith et al. [2] were the first to publish a shape that admits tilings of the plane, but never periodic tilings. This shape roughly has the form of a hat. Fig. 1 shows such hats. We assign numbers 1 to 6 for the unreflected tiles (top row). "1" is the hat in its basic orientation, and each rotation by 60 deg in counterclockwise direction increases its number by 1. To allow for a complete tiling of the plane, reflected tiles are needed. Hat "7" is a mirror image of hat "1" (mirrored with respect to the y-axis) and for each rotation in clockwise direction the assigned number increases by 1 (bottom row).

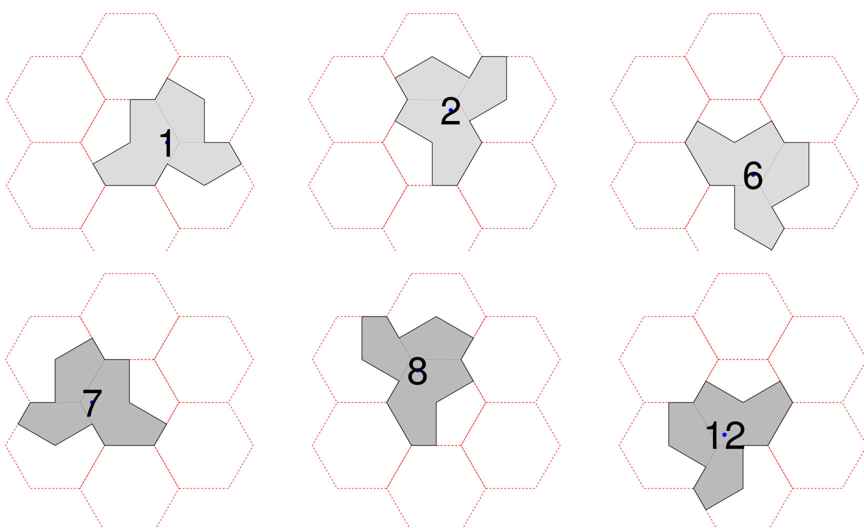

Figure 1: Monotiles in form of a hat that allow tilings of the plane, but never periodic tilings. There are 12 possible orientations.

2 Tesselation of the plane with hexagons

In order to be able to enumerate the possibilities to tile the plane with these 12 tiles, we tessellate the plane in hexagons. The hexagon "1" is the central hexagon. We continue the numbering of the hexagons with the hexagon to the upper right of the "1" and form a ring in counterclockwise direction up to the hexagon "7". We start counting the next ring with the "two o'clock" hexagon up to hexagon "19" and so forth. Fig. 2 illustrates the numbering of the hexagons.

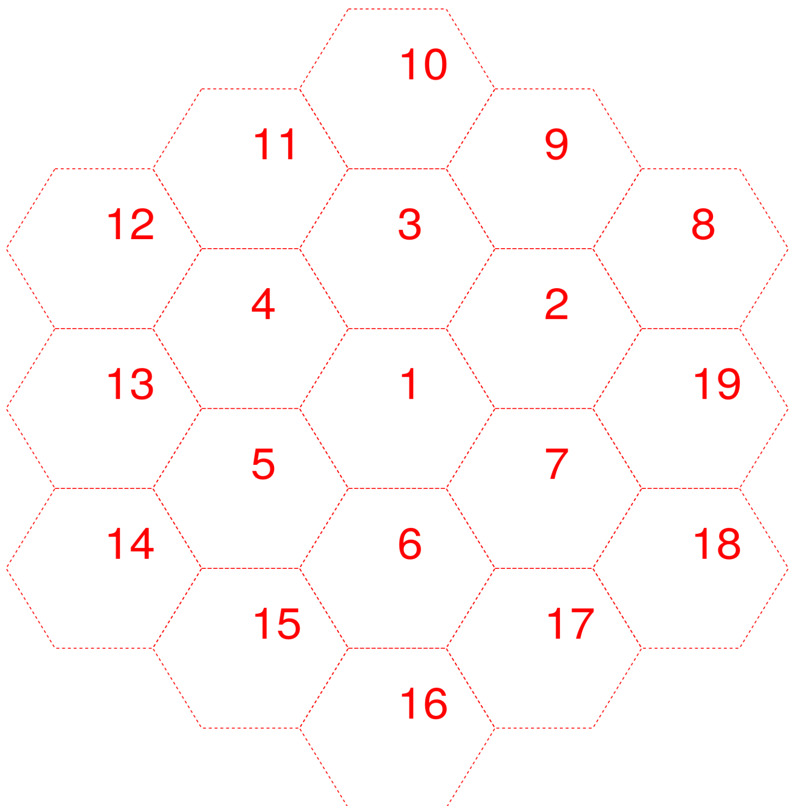

Figure 2: The plane is segmented by hexagons which are numbered counter-clockwise ring by ring.

A hexagon is assigned the number of a tile if the tile covers two thirds of the hexagon. Since the surface of a "hat" is 4/3 the surface of a hexagon, every fourth hexagon on average is covered by parts of three different tiles. In this case, we assign a zero to this hexagon.

3 Tiling attempts

Without loss of generality we place a tile "1" in the central hexagon like in Fig. 1.1 of the Smith paper [2]. Any other tiling can be transformed into this tiling by rotation or reflection. The hexagon '2' can be assigned hat "2", hat "10", or a zero. Fig. 3 shows the two hats that can be assigned to hexagon 2 and a third option where three different hats fill this hexagon (there are also other options for filling hexagon "2" with three hats). All other assignments will prevent correct plane tiling.

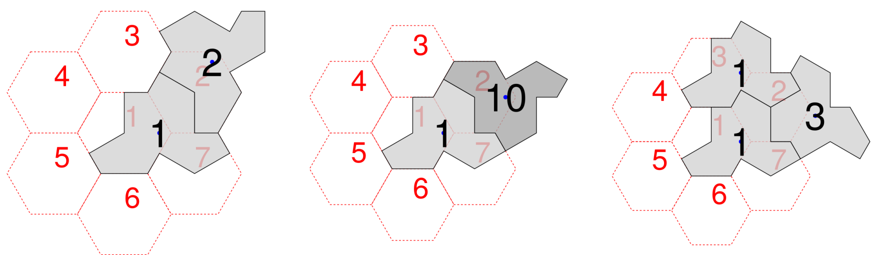

Figure 3: The three only possibilities to assign a number to hexagon "2" (in the right plot, where hexagon "2" is assigned a zero, hat "3" is assigned to hexagon "19").

For solution "1-2" (left plot of Fig. 3) four options exist for hexagon "3": assignment of hat "2", "3", "7" or "12" (top row of Fig. 4). For the solution "1-10" hat "12" needs to be assigned to hexagon "3". And for solution "1-0", hexagon "3" can be filled by the three hats "1", "4" and "8". All these eight possibilities are shown in Fig. 4.

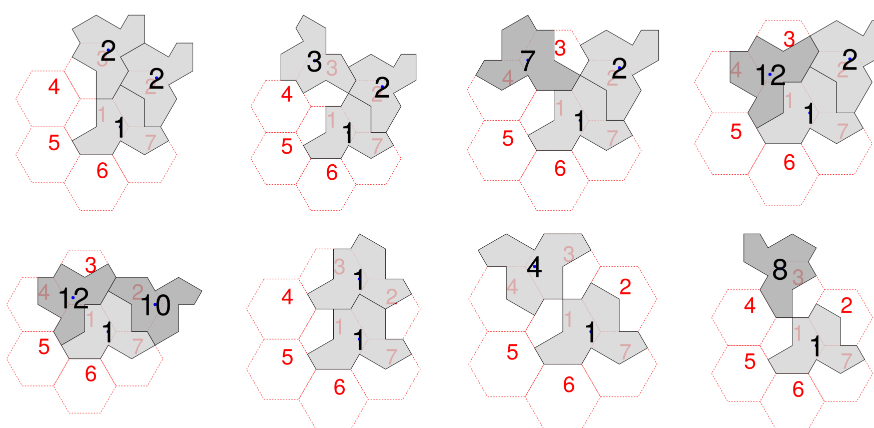

Figure 4: Eight possibilities to assign numbers to the first three hexagons (in the last three cases a zero is assigned to hexagon "2").

Up to hexagon 4 we found these 10 solutions:

- 1-0-1-0
- 1-0-1-3
- 1-0-1-6
- 1-0-4-0
- 1-0-8-6
- 1-2-2-6
- 1-2-3-10
- 1-2-7-6
- 1-2-12-5
- 1-10-12-0

Up to hexagon 7, i.e. the first complete ring, 21 solutions were found.

After each completion of a new ring, it is checked if the candidate solutions can be extended up to hexagon "91" (full ring 5) in such a way that the assignments of the hexagons in the outer ring are consistent with the 21 solutions that were found for the first ring. With this additional consistency check, brute force solutions like the one depicted in Fig. 5 are discarded right away. If you have a close look at hexagon "105", you can quickly notice that the two thirds of this hexagon cannot be filled correctly anymore.

If we denote the number of possible assignments for the first 19 hexagons which we found with our tests, this sequence of numbers is obtained: 1, 3, 8, 10, 12, 17, 21, 27, 28, 31, 33, 37, 40, 46, 48, 55, 56, 56, 56. It is sequence A377757 in OEIS ([1]).

4 Future work

It would be desirable to demand that the tiling can be continued to infinity, but this is much more difficult to prove. To decide whether a tiling is continuable, it is proposed to discard candidate solutions if it is not possible to add three further rings or to add two further rings with a consistency check of the outer ring.

An open question is, how the sequence A377757 can be approximated when n tends to infinity. If there is a linear relation, what is

$$\lim_{n\to\infty} \frac{a(n)}{n}?$$

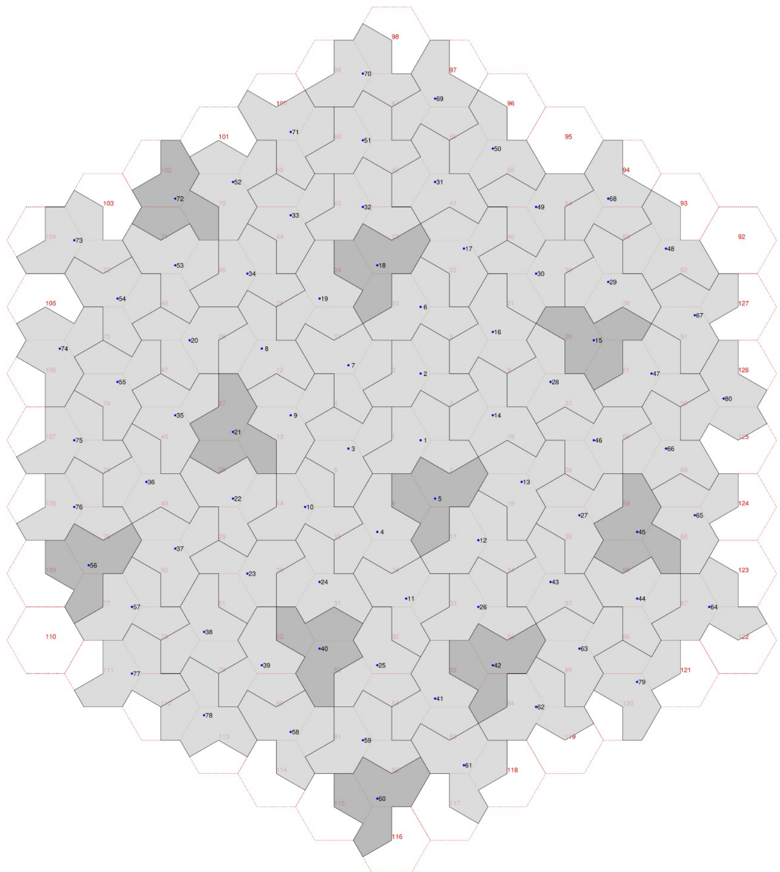

Figure 5: A brute force solution that is not continuable (see hexagon "105").

If it follows a power law, what is

$$\lim_{n\to\infty} a(n)^{\frac{1}{n}}?$$

The data obtained with our software is not sufficient to provide any reasonable answer.

References

[1] R. Jehn and K. Habermann. *Sequence A377757 of the On-Line Encyclopedia of Integer Sequences.* 2024.

[2] D. Smith, J. S. Myers, C. S. Kaplan, and C. Goodman-Strauss. *An aperiodic monotile.* 2023.