OEIS A372907

RICHARD J. MATHAR

Abstract. Convergence of sums of the form $\sum_{k\ge1}(\alpha^k\pm1)/(\beta^k\pm1)$ can be accelerated by resummation to sums of geometric sequences.

1. Parameters

Almost-geometric sequences in [1, A372907–A372911] have the format

$$\sum_{k\ge1}\frac{\alpha^k\pm1}{\beta^k\pm1} \tag{1}$$

for two constants $\alpha<\beta$ such that at large k the terms are close to geometric progressions. The standard manipulation to accelerate convergence is to reword the denominator such that well-known geometric series with closed form summations arise. Depending on the signs in numerator and denominator, four variants appear.

2. Denominator sign plus

$$\begin{aligned}
\sum_{k\ge0}\frac{\alpha^k\pm1}{\beta^k+1} &= \sum_{k\ge0}\frac{\alpha^k}{\beta^k+1}\pm\sum_{k\ge0}\frac{1}{\beta^k+1} = \frac12+\sum_{k\ge1}\frac{\alpha^k}{\beta^k+1}\pm[\alpha\to1]\\
&= \frac12+\sum_{k\ge1}\frac{\alpha^k}{\beta^k(1+\beta^{-k})}\pm[\alpha\to1]\\
&= \frac12+\sum_{k\ge1}\frac{\alpha^k}{\beta^k}\sum_{i\ge0}\frac{(-)^i}{\beta^{ik}}\pm[\alpha\to1]\\
&= \frac12+\sum_{i\ge0}(-)^i\sum_{k\ge1}(\frac{\alpha}{\beta^{i+1}})^k\pm[\alpha\to1]\\
&= \frac12+\sum_{i\ge0}(-)^i[\frac{1}{1-\frac{\alpha}{\beta^{i+1}}}-1]\pm[\alpha\to1] = \frac12+\sum_{i\ge0}(-)^i[\frac{-\alpha}{\alpha-\beta^{i+1}}]\pm[\alpha\to1]\\
&= \frac12+\sum_{i=0,2,4,\dots}\frac{-\alpha}{\alpha-\beta^{i+1}}-\sum_{i=1,3,5\dots}\frac{-\alpha}{\alpha-\beta^{i+1}}\pm[\alpha\to1]\\
&= \frac12+\sum_{m\ge0}\alpha\frac{\beta^{2m+1}(1-\beta)}{(\alpha-\beta^{2m+1})(a-\beta^{2m+2})}\pm[\alpha\to1].
\end{aligned} \tag{2}$$

The bracket $[\alpha\to1]$ means all previous terms with α substituted by 1 reappear.

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3. Denominator sign minus

For β^k-1 in the denomiator the sums starts at $k=1$ to avoid division by zero:

$$\sum_{k\ge 1}\frac{\alpha^k\pm 1}{\beta^k-1}=\sum_{k\ge 1}\frac{\alpha^k}{\beta^k-1}\pm\sum_{k\ge 1}\frac{1}{\beta^k-1}=\sum_{k\ge 1}\frac{\alpha^k}{\beta^k-1}\pm[\alpha\to 1] \tag{3}$$

$$=\sum_{k\ge 1}\frac{\alpha^k}{\beta^k(1-\beta^{-k})}\pm[\alpha\to 1]$$

$$=\sum_{k\ge 1}\frac{\alpha^k}{\beta^k}\sum_{i\ge 0}\frac{1}{\beta^{ik}}\pm[\alpha\to 1]$$

$$=\sum_{i\ge 0}\sum_{k\ge 1}(\frac{\alpha}{\beta^{i+1}})^k\pm[\alpha\to 1]$$

$$=\sum_{i\ge 0}[\frac{1}{1-\frac{\alpha}{\beta^{i+1}}}-1]\pm[\alpha\to 1]=-\alpha\sum_{i\ge 0}\frac{1}{\alpha-\beta^{i+1}}\pm[\alpha\to 1].$$

Appendix A. Maple implementation

```
#!/usr/bin/env maple
interface(quiet=true):

Digits := 50 :

# sum_{k>=0,1} (a^k +siga)/(b^k+sigb)
# Direct naive summation of the first 40 terms
Galmost := proc(a,b,siga,sigb)
        local val,m ;
        val := 0.0 ;
        for m from 0 to 40 do
                val := val+evalf( (a^m+siga)/(b^m+sigb)) ;
                print(m,%) ;
        end do:
end proc:
# Galmost(2,3,1,1) ;

# sum_{k>=0,1} (a^k +siga)/(b^k+sigb)
# where siga and sigb are +1 or -1.
# Resummed summation of the first 20 terms
GalmostResum := proc(a,b,siga,sigb)
        local val,m ;
        if sigb = 1 then
                val := 1/2*(1+siga) ;
                for m from 0 to 20 do
                        a*b^(2*m+1)*(b-1) ;
                        %/(a-b^(2*m+1)) / (a-b^(2*m+2)) ;
                        val := val+evalf(%) ;
                        b^(2*m+1)*(b-1) ;
                        %/(1-b^(2*m+1)) / (1-b^(2*m+2)) ;
```

```
                        val := val+siga*evalf(%) ;
                        print(m,%) ;
                end do:
        elif sigb = -1 then
                val := 0.0 ;
                for m from 0 to 20 do
                        -a/(a-b^(m+1))  ;
                        val := val+evalf(%) ;
                        -1/(1-b^(m+1))  ;
                        val := val+siga*evalf(%) ;
                        print(m,%) ;
                end do:
        end if;
end proc:

# evaluate A372907
GalmostResum(2,3,-1,-1) ;

# evaluate A372908
GalmostResum(2,3,1,1) ;

# evaluate A372909
GalmostResum(2,4,1,1) ;

# evaluate A372910
GalmostResum(3,6,1,1) ;

# evaluate A372911
GalmostResum(4,8,1,1) ;
```

References

1. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2024), https://oeis.org/. MR 3822822
   URL: http://www.mpia.de/~mathar

Hoeschstr. 7, 52372 Kreuzau, Germany