ON AN INTEGER SEQUENCE RELATED TO EULER'S FORMULA FOR THE STIRLING NUMBERS OF THE SECOND KIND

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Let k, m and n be three nonnegative integers. We define

$$a_{m,n,k} = \sum_{i=0}^{m} \frac{(-1)^i}{m!} \binom{m}{i} (k+m-i)^{n+m}.$$

It may be shown that the following recursion holds:

$$a_{m,n,k} = (k+m)a_{m,n-1,k} + a_{m-1,n,k}, \tag{1}$$

with initial values $a_{0,n,k} = k^n$ and $a_{m,0,k} = 1$.

For $n \geq 0$ and $0 \leq k \leq n$ let $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the second kind. Recall that (e.g., [1, (6.3)])

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}.$$

We prove the following result, establishing two of the three conjectures stated in A372118 [3] and correcting the third conjecture.

Theorem 1. *(1) For fixed m and k, the generating function $A_{m,k}(x)$ for the numbers $a_{m,n,k}$ is given by*

$$A_{m,k}(x) = \prod_{i=k}^{k+m} \frac{1}{1-ix}.$$

(2) For fixed m and n, the exponential generating function $B_{m,n}(x)$ for the numbers $a_{m,n,k}$ is given by

$$B_{m,n}(x) = e^x \sum_{k=0}^{n} \binom{k+m}{m} \left\{ {n+m \atop k+m} \right\} x^k.$$

(3) We have

$$a_{m,n,k} = \sum_{i=0}^{n} \frac{(i+m)!}{m!} \binom{k}{i} \left\{ {n+m \atop i+m} \right\}.$$

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Proof. (1) Multiplying (1) by x^n, summing over $n \geq 1$ and adding $a_{m,0,k} = a_{m-1,0,k} = 0$ to both sides, we obtain

$$\sum_{n\geq 0} a_{m,n,k}x^n = (k+m)\sum_{n\geq 1} a_{m,n-1,k}x^n + \sum_{n\geq 0} a_{m-1,n,k}x^n.$$

Writing this equation in terms of the generating functions, and solving for $A_{m,k}(x)$, we have

$$A_{m,k}(x) = \frac{1}{1-(k+m)x}A_{m-1,k}(x),$$

from which the assertion immediately follows by induction.

(2) Multiplying (1) by $x^k/k!$ and summing over $k \geq 0$, we obtain

$$\sum_{k\geq 0}\frac{a_{m,n,k}}{k!}x^k = x\sum_{k\geq 1}\frac{a_{m,n-1,k}}{(k-1)!}x^{k-1} + m\sum_{k\geq 0}\frac{a_{m,n-1,k}}{k!}x^k + \sum_{k\geq 0}\frac{a_{m-1,n,k}}{k!}x^k.$$

Writing this equation in terms of the generating functions, we have

$$B_{m,n}(x) = x\frac{d}{dx}B_{m,n-1}(x) + mB_{m,n-1}(x) + B_{m-1,n}(x).$$

Assume that the assertion holds for every $m, n \geq 1$. Thus,

$$\begin{aligned}
B_{m,n}(x) &= x\frac{d}{dx}\left(e^x\sum_{k=0}^{n-1}\binom{k+m}{m}\left\{ {n+m-1 \atop k+m} \right\}x^k\right) \\
&\quad + me^x\sum_{k=0}^{n-1}\binom{k+m}{m}\left\{ {n+m-1 \atop k+m} \right\}x^k + e^x\sum_{k=0}^{n}\binom{k+m-1}{m-1}\left\{ {n+m-1 \atop k+m-1} \right\}x^k \\
&= e^x\left(\binom{n+m-1}{m} + \binom{n+m-1}{m-1}\right)x^n \\
&\quad + e^x\sum_{k=0}^{n-1}\Bigg[\left(\binom{k+m-1}{m} + \binom{k+m-1}{m-1}\right)\left\{ {n+m-1 \atop k+m-1} \right\} \\
&\qquad\qquad + (k+m)\binom{k+m}{m}\left\{ {n+m-1 \atop k+m} \right\}\Bigg]x^k \\
&= e^x\binom{n+m}{m}x^n + e^x\sum_{k=0}^{n-1}\binom{k+m}{m}\left((k+m)\left\{ {n+m-1 \atop k+m} \right\} + \left\{ {n+m-1 \atop k+m-1} \right\}\right)x^k \\
&= e^x\sum_{k=0}^{n}\binom{k+m}{m}\left\{ {n+m \atop k+m} \right\}x^k.
\end{aligned}$$

(3) The identity is a special case of [2, (9.26)].

$\square$

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics - A Foundation for Computer Science*, 1989.

[2] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers: The Unpublished Notes of H W Gould*, World Scientific, 2015.

[3] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.