On a problem related to the integer lattice and its layers

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Let k and n be two nonnegative integers, and let m be a natural number. Denote by $a_{n,k}^{(m)}$ the number of integer points $(x_1,\ldots,x_m)$, such that $\max_{1\leq i\leq m}|x_i|\leq k$ and $\sum_{i=1}^m |x_i| = n$. For fixed k and m, denote by $A_k^{(m)}(x)$ the generating function for the numbers $a_{n,k}^{(m)}$.

The purpose of the note is to prove the following result, which we shall use to prove a conjecture stated in A371835 [1].

Theorem 1. *We have*

$$A_k^{(m)}(x) = \left(1+2\sum_{i=1}^k x^i\right)^m. \tag{1}$$

Proof. For $m=1$, we have

$$a_{n,k}^{(1)} = \begin{cases} 1, & \text{if } n=0; \\ 2, & \text{if } 1\leq n\leq k; \\ 0, & \text{otherwise.}\end{cases}$$

Thus, (1) holds in this case. Assuming that (1) holds for $m\geq 1$, we have

$$a_{n,k}^{(m+1)} = \sum_{i=0}^{m+1} 2^i\binom{m+1}{i} a_{n-ik,k-1}^{(m+1-i)}.$$

Multiplying both sides of this equation by x^n and summing over $n\geq 0$, we obtain

$$A_k^{(m+1)}(x) = \sum_{i=0}^{m+1}(2x^k)^i\binom{m+1}{i}A_{k-1}^{(m+1-i)}(x)$$

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$$= \sum_{i=0}^{m+1} (2x^k)^i \binom{m+1}{i} \left(1 + 2\sum_{j=1}^{k-1} x^j\right)^{m+1-i}. \qquad (2)$$

By the binomial theorem the right-hand side of (2) is equal to

$$\left(1 + 2\sum_{j=1}^{k-1} x^j + 2x^k\right)^{m+1} = \left(1 + 2\sum_{j=1}^{k} x^j\right)^{m+1},$$

and the proof is complete. □

Denote by $b_{n,k}^{(m)}$ the number of integer points $(x_1, \ldots, x_m)$, such that $\max_{1\le i\le m} |x_i| \le k$ and $\sum_{i=1}^{m} |x_i| \le n$. For fixed k and m, denote by $B_k^{(m)}(x)$ the generating function for the numbers $b_{n,k}^{(m)}$. Clearly, $B_k^{(m)}(x) = A_k^{(m)}(x)/(1-x)$. The following result extends and validates the conjecture in A371835.

Corollary 2. *For $m = 3$ we have*

$$B_k^{(3)}(x) = \frac{\left(1 + 2\sum_{i=1}^{k} x^i\right)^3}{1-x}.$$

In particular,

$$b_{n,k}^{(3)} = \frac{1}{3}\begin{cases} 4n^3 + 6n^2 + 8n + 3, & \text{if } 0 \le n < k; \\ 12k^3 - 36k^2n + 36kn^2 - 8n^3 + 6n^2 + 6k + 2n + 3, & \text{if } k \le n < 2k; \\ -84k^3 + 108k^2n - 36kn^2 + 4n^3 - 72k^2 + 72nk - 12n^2 - 6k + 8n + 3, & \text{if } 2k \le n < 3k; \\ 24k^3 + 36k^2 + 18k + 3, & \text{if } n \ge 3k. \end{cases}$$

Proof. We have

$$\begin{aligned} B_k^{(3)}(x) &= \frac{\left(1 + 2\sum_{i=1}^{k} x^i\right)^3}{1-x} \\ &= \frac{1}{1-x}\sum_{j=0}^{3}\binom{3}{j}\left(2\sum_{i=1}^{k} x^i\right)^j \\ &= \sum_{j=0}^{3}\binom{3}{j}(2x)^j \frac{(1-x^k)^j}{(1-x)^{j+1}} \end{aligned}$$

$$= \frac{-8x^{3k+3} + 12x^{2k+3} + 12x^{2k+2} - 6x^{k+3} - 12x^{k+2} + x^3 - 6x^{k+1} + 3x^2 + 3x + 1}{(1-x)^4}$$

$$= \sum_{n \geq 0} \binom{n+3}{n} \Big(- 8x^{n+3k+3} + 12x^{n+2k+3} + 12x^{n+2k+2} - 6x^{n+k+3} - 12x^{n+k+2} + x^{n+3} - 6x^{n+k+1} + 3x^{n+2} + 3x^{n+1} + x^n \Big).$$

It follows that, for $0 \leq n < k$, we have

$$b_{n,k}^{(3)} = \binom{n}{3} + 3\binom{n+1}{3} + 3\binom{n+2}{3} + \binom{n+3}{3} = \frac{4n^3 + 6n^2 + 8n + 3}{3}.$$

For $k \leq n < 2k$, we have

$$b_{n,k}^{(3)} = -6\binom{n-k}{3} - 12\binom{n-k+1}{3} + \binom{n}{3} - 6\binom{n-k+2}{3} + 3\binom{n+1}{3} + 3\binom{n+2}{3} + \binom{n+3}{3} = \frac{12k^3 - 36k^2n + 36kn^2 - 8n^3 + 6n^2 + 6k + 2n + 3}{3}.$$

For $2k \leq n < 3k$, we have

$$b_{n,k}^{(3)} = 12\binom{n-2k}{3} + 12\binom{n-2k+1}{3} - 6\binom{n-k}{3} - 12\binom{n-k+1}{3} + \binom{n}{3} - 6\binom{n-k+2}{3} + 3\binom{n+1}{3} + 3\binom{n+2}{3} + \binom{n+3}{3} = \frac{-84k^3 + 108k^2n - 36kn^2 + 4n^3 - 72k^2 + 72nk - 12n^2 - 6k + 8n + 3}{3}.$$

Finally, for $n \geq 3k$, we have

$$b_{n,k}^{(3)} = -8\binom{n-3k}{3} + 12\binom{n-2k}{3} + 12\binom{n-2k+1}{3} - 6\binom{n-k}{3} - 12\binom{n-k+1}{3} + \binom{n}{3} - 6\binom{n-k+2}{3} + 3\binom{n+1}{3} + 3\binom{n+2}{3} + \binom{n+3}{3} = 8k^3 + 12k^2 + 6k + 1. \qquad \square$$

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., https://oeis.org.

[2] B. Bajnok, Additive Combinatorics: A Menu of Research Problems, Chapman and Hall/CRC, 2018.