Formulas for A368548 and A375783

Chai Wah Wu

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Let $a(n)$ denote the terms of OEIS A368548, the number of palindromic partitions of n.

Theorem 1. *$a(n) = x + y$ where*

$$x = \begin{cases} 0 & , n \text{ even} \\ \sum_{d|\frac{n+1}{2}} \binom{d-2+\frac{n+1}{2d}}{d-1} & , n \text{ odd} \end{cases}$$

and $y = 2\sum_{d|n+1, d\geq 3, d \text{ is odd}} \binom{\frac{d-5}{2}+\frac{n+1}{d}}{\frac{d-3}{2}}$.

Proof. From the generating function in Hemmer and Westrem [1] (Theorem 3.1) to find $a(n)$ we need to solve the equations $2kl+2k+2l+1 = n$ and $2kl+2k+3l+2 = n$. The first equation reduces to $2(k+1)(l+1) = n+1$ which has no solutions if n is even. If n is odd, $(k+1)(l+1) = \frac{n+1}{2}$ and we set $k+1 = d$, $l+1 = \frac{n+1}{2d}$ for each divisor d of $\frac{n+1}{2}$ and $\binom{k+l}{k} = \binom{d-2+\frac{n+1}{2d}}{d-1}$. This leads to the term x. The second equation reduces to $(2k+3)(l+1) = n+1$. Note that $2k+3$ is odd and we set $2k+3$ to be an odd divisor $d \geq 3$ of $n+1$. Then $l+1 = \frac{n+1}{d}$ and $2\binom{k+l}{k} = 2\binom{\frac{d-5}{2}+\frac{n+1}{d}}{\frac{d-3}{2}}$. □

Corollary 1. *If $n > 1$ and $n+1$ is prime, then $a(n) = 2$.*

Proof. Since $n > 1$, $n+1 = p$ being prime implies n is even, i.e., $x = 0$ in the Theorem above. The only odd divisor ≥ 3 of $n+1$ is p and $y = 2\binom{\frac{p-5}{2}+1}{\frac{p-3}{2}} = 2\binom{\frac{p-3}{2}}{\frac{p-3}{2}} = 2$, i.e., $a(n) = 2$. □

Corollary 1 also follows from Theorem 3.3 in Hemmer and Westrem [1].

Corollary 2. *If $n > 3$ is odd and $\frac{n+1}{2}$ is prime, then $a(n) = \frac{n+3}{2}$.*

Proof. Since $n > 3$, this means that $\frac{n+1}{2} = p$ is an odd prime. For x, the only divisors of $\frac{n+1}{2}$ are 1 and p and $x = 2\binom{p-2+1}{p-1} = 2$. Similarly, the only odd divisor ≥ 3 of $n+1$ is p and $y = 2\binom{\frac{p-5}{2}+2}{\frac{p-3}{2}} = 2\binom{\frac{p-1}{2}}{1} = p-1$. Thus $a(n) = x + y = p + 1 = \frac{n+3}{2}$. □

Corollary 3. $a(2^n - 1) = \sum_{i=0}^{n-1} \binom{2^i+2^{n-i-1}-2}{2^i-1}$.

Proof. Since 2^n is either even or < 3, this implies that $y = 0$. The result then follows since the divisors of 2^{n-1} are 2^i, for $i = 0, 1, \cdots, n-1$. □

A similar derivation shows that $R(n)$ (OEIS A375783) in Table 5.1 in Hemmer and Westrem [1] has a similar formula.

Theorem 2. *Let $T(n,k)$ be the table in OEIS A183917. Then $R(n) = x + y$ where*

$$x = \begin{cases} 0 & , n \text{ even} \\ \sum_{d|\frac{n+1}{2}} T(\frac{n+1}{d} - 2, d - 1) & , n \text{ odd} \end{cases}$$

and $y = 2\sum_{d|n+1, d\geq 3, d \text{ is odd}} T(d-2, \frac{n+1}{d} - 1)$.

Corollary 4. *If $n > 1$ and $n + 1$ is prime, then $R(n) = a(n) = 2$.*

Proof. The same argument as Corollary 1 shows that $R(n) = y = 2T(p - 2, 1) = 2$. ☐

Corollary 5. *If $n > 3$ is odd and $\frac{n+1}{2}$ is prime, then $R(n) = a(n) = \frac{n+3}{2}$.*

Proof. $x = T(n-1, 0)+T(0, p-1) = 1+1 = 2$. $y = 2T(p-2, 1) = 2\frac{p-1}{2} = p-1$. Thus $R(n) = p+1 = \frac{n+3}{2}$. ☐

Corollary 6. $R(2^n - 1) = \sum_{i=0}^{n-1} T(2^{n-i} - 2, 2^i - 1)$.

References

[1] David J. Hemmer and Karlee J. Westrem, "Palindrome Partitions and the Calkin-Wilf Tree" arXiv:2402.02250, 2024.