For PIE we need to choose the required poset which consists of nodes Q_S where S is a set of disjoint m-cycles chosen from the m-cycles that can be formed using the elements of $[n]$. We build rows for fixed cardinality of $|S|$. We consider the poset spanned by the nodes on row k and the top row, i.e. the row where $|S| = \lfloor n/m \rfloor$. The weight attached to the node Q_S is $(-1)^{|S|-k}\binom{|S|}{k}$ and the poset is ordered by subset inclusion of S. The permutations that are represented at each node consist of the $|S|$ cycles of length m with the rest being arranged at liberty. The cardinality of the permutations represented at a node Q_S is thus $(n-m|S|)!$. We now count the permutations represented by the nodes of the subposet according to their weight. Do this in two ways: the intersection with row p contains

$$\binom{n}{m, m, \ldots, m, n-pm} \frac{1}{p!}(m-1)!^p = \frac{n!}{m^p p!} \frac{1}{(n-pm)!}.$$

nodes. The weight on these is $(-1)^{p-k}\binom{p}{k}$ and the cardinality of the set of permutations being represented is $(n-pm)!$ for a total of

$$\sum_{p=k}^{\lfloor n/m \rfloor} (-1)^{p-k} \binom{p}{k} \frac{n!}{m^p p!}$$

$$= n! \sum_{p=0}^{\lfloor n/m \rfloor - k} (-1)^p \binom{p+k}{k} \frac{1}{m^{p+k}(p+k)!}$$

$$= \frac{n!}{m^k k!} \sum_{p=0}^{\lfloor n/m \rfloor - k} \frac{(-1)^p}{m^p p!}.$$

Good, this is our claim. Now to count in the second way, what is the total weight on a permutation with precisely a set P of m-cycles where $k \le |P| \le \lfloor n/m \rfloor$. It is represented at all Q_S where $S \subseteq P$ and $|S| \ge k$ giving the sum

$$\sum_{S \subseteq P, |S| \ge k} (-1)^{|S|-k} \binom{|S|}{k} = \sum_{p=k}^{|P|} \binom{|P|}{p} (-1)^{p-k} \binom{p}{k}.$$

Now a permutation with $|P| = k$ i.e. exactly k m-cycles therefore contributes with weight one, as desired. For $|P| > k$ we find

$$\sum_{p=k}^{|P|} (-1)^{p-k} \frac{|P|!}{(|P|-p)! \times k! \times (p-k)!}$$

$$= \binom{|P|}{k} \sum_{p=k}^{|P|} (-1)^{p-k} \binom{|P|-k}{p-k}$$

$$= \binom{|P|}{k} \sum_{p=0}^{|P|-k} (-1)^p \binom{|P|-k}{p} = \binom{|P|}{k} (1-1)^{|P|-k} = 0.$$

We see that permutations with more than k m-cycles contribute with weight zero, which concludes the PIE argument.