Using combinatorial classes as in *Analytic Combinatorics* by Flajolet and Sedgewick we have the following class $\mathcal{P}$ of permutations with **exactly** $k$ $m$-cycles

$$\mathcal{P} = \textsc{Set}(\textsc{Cyc}_{=1}(\mathcal{Z}) + \textsc{Cyc}_{=2}(\mathcal{Z}) + \cdots \\ + \textsc{Cyc}_{=m-1}(\mathcal{Z}) + \textsc{Cyc}_{=m+1}(\mathcal{Z}) + \cdots) \times \textsc{Set}_{=k}(\textsc{Cyc}_{=m}(\mathcal{Z})).$$

This gives the EGF

$$\begin{aligned} G(z) &= \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^{m-1}}{m-1} + \frac{z^{m+1}}{m+1} + \cdots\right) \frac{1}{k!} \left[\frac{z^m}{m}\right]^k \\ &= \exp\left(\log\frac{1}{1-z}\right) \exp\left(-\frac{z^m}{m}\right) \frac{1}{k!} \left[\frac{z^m}{m}\right]^k \\ &= \frac{1}{1-z} \exp\left(-\frac{z^m}{m}\right) \frac{1}{k!} \left[\frac{z^m}{m}\right]^k. \end{aligned}$$

Extracting the coefficient on the EGF in $z$ we find

$$\begin{aligned} n![z^n]G(z) &= n![z^n] \frac{1}{1-z} \exp\left(-\frac{z^m}{m}\right) \frac{1}{k!} \left[\frac{z^m}{m}\right]^k \\ &= \frac{n!}{m^k k!} [z^{n-mk}] \frac{1}{1-z} \exp\left(-\frac{z^m}{m}\right) \\ &= \frac{n!}{m^k k!} \sum_{q=0}^{\lfloor n/m \rfloor - k} [z^{n-mk-qm}] \frac{1}{1-z} [z^{qm}] \exp\left(-\frac{z^m}{m}\right) \\ &= \frac{n!}{m^k k!} \sum_{q=0}^{\lfloor n/m \rfloor - k} [z^{qm}] \exp\left(-\frac{z^m}{m}\right) \\ &= \frac{n!}{m^k k!} \sum_{q=0}^{\lfloor n/m \rfloor - k} [z^q] \exp\left(-\frac{z}{m}\right). \end{aligned}$$

This is

$$\boxed{\frac{n!}{m^k k!} \sum_{q=0}^{\lfloor n/m \rfloor - k} \frac{(-1)^q}{m^q q!} \underset{n\to\infty}{\sim} \frac{n!}{m^k k!} \exp(-1/m).}$$

Observe that when $n = mk$ so there are no other cycles this will produce $\frac{n!}{m^k k!}$. Note also that e.g. it is not possible to have exactly $n-1$ fixed points in a permutation of length $n$ because the last fixed point is forced to join. And indeed setting $m = 1$ we get (derangement numbers appear)

$$\frac{n!}{k!} \sum_{q=0}^{n-k} \frac{(-1)^q}{q!} = \frac{n!}{(n-1)!} \sum_{q=0}^{1} \frac{(-1)^q}{q!} = n \times (1-1) = 0.$$