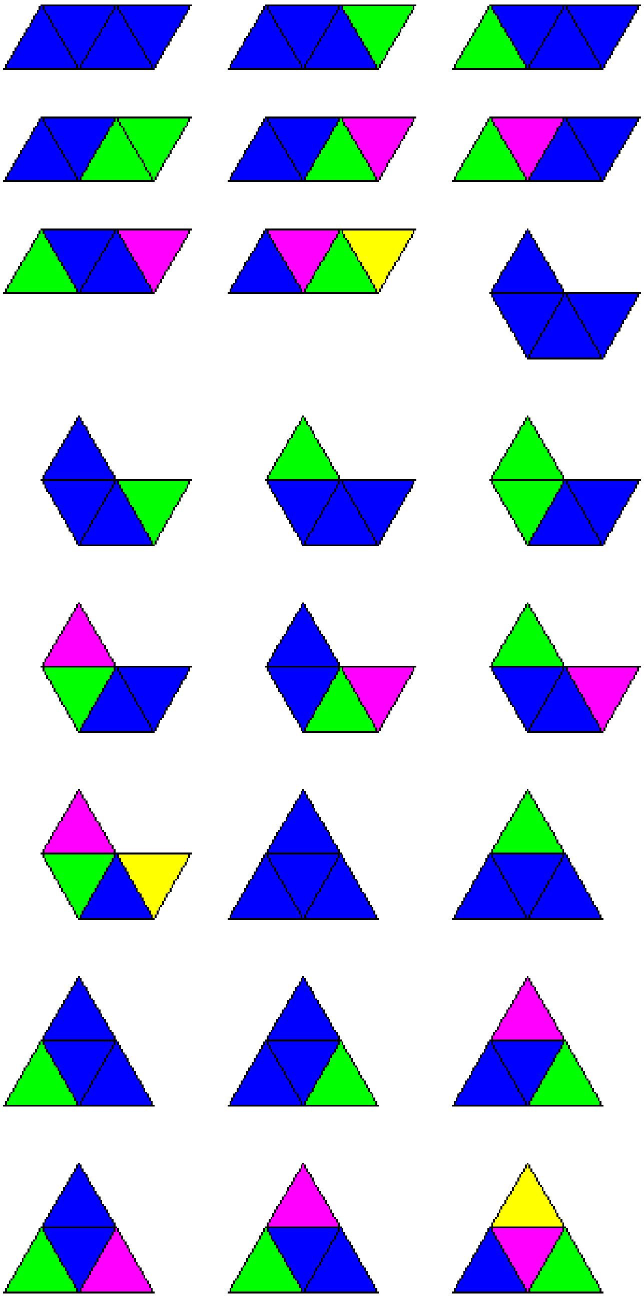
Polyiamond tiling – Version 2 April 2024

Consider the sequence a(n) = the maximum number of distinct tilings of a polyiamond of size n using any combination of polyiamond tiles of sizes 1 through n.

DATA: 1, 2, 4, 8, 16, 58, 116, 232, 464, 1690, 3380, 6760, 24712, 49424, 98848, 361258, … (see published sequence)

The sequence considers reflections and rotations as distinct tilings. The polyiamonds being tiled and the tiles themselves may be with or without holes.

The following diagram shows the 8 distinct tilings of each of the 3 tetriamonds. For example, in the first line we see tilings made of (i) the tetriamond itself, (ii) one triamond + one moniamond, and (iii) one moniamond + one triamond.



The following table shows, for polyiamond sizes up to 17, what numbers of distinct tiling patterns are possible. E.g., for size 6, 11 free polyiamonds have 32 distinct patterns, and 1 has 58 distinct patterns. The trailing numbers are the values of the current sequence.

Underneath each pair of numbers are the values of perimeter, internal points[1] count, internal edges count and totally internal edge count.

1	1 (1) 3:0:0:0														
2	2 (1) 4:0:1:0														
3	4 (1) 5:0:2:0														
4	8 (3) 6:0:3:0														
5	16 (4) 7:0:4:0														
6	32 (11) 8:0:5:0	58 (1) 6:1:6:0													
7	64 (23) 9:0:6:0	116 (1) 7:1:7:0													
8	128 (62) 10:0:7:0	232 (4) 8:1:8:0													
9	256 (149) 11:0:8:0	464 (11) 9:1:9:0													
10	512 (409) 12:0:9:0	928 (38) 10:1:10:0	1690 (1) 8:2:11:1												
11	1024 (1066) 13:0:10:0	1856 (118) 11:1:11:0	3380 (2) 9:2:12:1												
12	2048 (2931) 14:0:11:0	3712 (387) 12:1:12:0	4084 (1) 12:0:12:0	6728 (1) 10:2:13:0	6760 (14) 10:2:13:1										

[1] See definitions in appendix

13	4096 (7981) 15:0:12:0	7424 (1197) 13:1:13:0	8168 (2) 13:0:13:0	13456 (4) 11:2:14:0	13520 (50) 11:2:14:1	24712 (1) 9:3:15:3								
14	8192 (22166) 16:0:13:0	14848 (3751) 14:1:14:0	16336 (11) 14:0:14:0	16370 (1) 14:0:14:0	26912 (24) 12:2:15:0	27040 (209) 12:2:15:1	49234 (2) 10:3:16:2	49424 (2) 10:3:16:3						
15	16384 (61508) 17:0:14:0	29696 (11563) 15:1:15:0	32672 (47) 15:0:15:0	32740 (3) 15:0:15:0	53824 (110) 13:2:16:0	54080 (732) 13:2:16:1	59228 (1) 13:1:16:0	98468 (8) 11:3:17:2	98848 (11) 11:3:17:3					
16	32768 (172267) 18:0:15:0	59392 (35636) 16:1:16:0	65344 (189) 16:0:16:0	65480 (20) 16:0:16:0	65520 (1) 16:0:16:0	107648 (508) 14:2:17:0	108160 (2566) 14:2:17:1	118456 (7) 14:1:17:0	196040 (2) 12:3:18:1	196936 (53) 12:3:18:2	197696 (47) 12:3:18:3	361258 (1) 10:4:19:5		
17	65536 (483088) 19:0:16:0	118784 (109142) 17:1:17:0	130688 (706) 17:0:17:0	130960 (86) 17:0:17:0	131040 (7) 17:0:17:0	215296 (2106) 15:2:18:0	216320 (8467) 15:2:18:1	236912 (46) 15:1:18:0	237378 (1) 15:1:18:0	392080 (25) 13:3:19:1	393872 (240) 13:3:19:2	395392 (189) 13:3:19:3	719824 (1) 11:4:20:4	722516 (3) 11:4:20:5

There are various patterns apparent in the table:

a) The first data column has values equal to 2^(n-1).

b) If the value x is in line n, then 2*x appears in line n+1.

c) a(n+1) is often but not always 2*a(n). In particular, consider the sequence A067628 (minimal perimeter for a polyiamond of size n); a(n+1) is close to 3.65 a(n) each time A067628(n+1) < A067628(n), and is equal to 2a(n) otherwise.

d) For any given size of polyiamond, a smaller perimeter implies a larger number of tilings; for any given pair of values of size and perimeter, a smaller number of internal completely surrounded points implies a larger number of tilings.

e) For any given size of polyiamond, it is remarkable how few different numbers of tilings are generated. For example, 3334 size 12 polyiamonds have just 5 different numbers of tilings.

Of these, (a) and (b) depend directly on the following Theorems 1 to 4.

Definition: MTP : maximally tilable polyiamond

Definition: B(P) is the number of cells that form part of branches of a polyiamond

Definition: the core of a non-treelike polyiamond is what remains after the removal of branches

Theorem 1: If T(P) is the number of tiling patterns for some polyiamond P, and Q is a new polyiamond formed by adding one triangle to P such that the said triangle touches only one other, then T(Q) = 2*T(P). No polyiamond exists for which the addition of a triangle that touches only one other is impossible.

Theorem 2: As a consequence of (1), T(P) = 2^(n-1) for any treelike[2] polyiamond of size n (the first column of the above table).

Theorem 3: For any non-treelike polyiamond P of size n, T(P) > 2^(n-1).

Theorem 4: For any given size, n, of polyiamond, and value t, that is the number of tilings of some polyiamond of size n, then for any integer k >= 1 there exist polyiamonds of size n+k that have 2^kt tilings.

[2] See definition in appendix

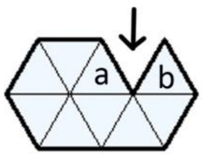
Note also:

Theorem 5: If T(P) is the number of tiling patterns for some polyiamond P, and Q is a new polyiamond formed by adding one triangle to P such that said triangle touches precisely two others, then:

$$3*T(P) < T(Q) < 4*T(P)$$

Proof:

Consider the addition of a triangle that touches both triangles a and b of the existing polyiamond:

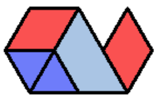


We want to determine the number of tilings when the triangle is added. Say T_a is some tile that contains triangle *a*, and T_b is some tile that contains triangle *b*.
In principle there are 4 cases:
1) The added triangle itself is considered as a new, standalone tile.
2) The added triangle builds a new tile together with tile T_a.
3) The added triangle builds a new tile together with tile T_b.
4) The added triangle builds a new tile together with the tiles T_a and T_b.

Define also, with respect to the original polyiamond P:

N_A is the number of tilings where *a* and *b* belong to the same tile.

N_B is the number of tilings where *a* and *b* belong to different tiles, but these two tiles have a common edge.



N_C is the number of all remaining tilings.

So if T(P) is the number of tilings of P, then $T(P) = N_A + N_B + N_C$

In each case, how may new tilings result from the addition of the triangle?

A: for each of the N_A tilings, cases 1 & 4 apply, so there are $2N_A$ new tilings in Q.

B: for each of the N_B tilings, cases 1, 2 & 3 apply so there are $3N_B$ new tilings in Q.

C: for each of the N_C tilings, all cases apply so there are $4N_C$ new tilings in Q.

Therefore (**Formula 1**), the total number of new tilings is: $T(Q) = 2N_A + 3N_B + 4N_C$

The minimal n for P is 5. Otherwise, the added triangle could touch two existing triangles.
$N_A > 0$, as certainly the tiling that consists just of the tile P counts towards N_A.
Case B can be constructed by splitting a tile that contains both triangles into two tiles, where one is connected with triangle a and the other with triangle b. There are at least 4 possibilities to do this. Therefore $N_B >= 4N_A$.
In each tiling of case B a tile that contains triangle a or b consists of at least 3 triangles and can be split into 2 or 3 tiles. Thus $N_C > N_B$.
Summary: $0 < N_A < N_B < N_C$
$3*T(P) = 3 N_A + 3 N_B + 3 N_C < 3 N_A + 3 N_B + 3 N_C + (N_C - N_A) = 2 N_A + 3 N_B + 4 N_C = T(Q)$
Therefore $3*T(P) < T(Q)$
$4*T(P) = 4 N_A + 4 N_B + 4 N_C > 4 N_A + 4 N_B + 4 N_C - (2 N_A + N_B) = 2 N_A + 3 N_B + 4 N_C = T(Q)$
Therefore $4*T(P) > T(Q)$

Corollary to Theorem 5: If P is an “almost-ring” snake[3] of size n and Q is a ring formed by adding one triangle to P, then T(Q) / T(P) is less than 4 but arbitrarily close to 4.

Proof:
Recall that $T(P) = 2^{n-1}$.
With respect to P, it is easy to see that:

$N_A = 1$

$N_B = n – 1$

$N_C = 2^{n-1} – n$

By Formula 1:

$T(Q) = 2N_A + 3N_B + 4N_C = 2 + 3n – 3 + 4(2^{n-1} – n) = 4.2^{n-1} - n – 1$

Therefore:

$T(Q) / T(P) = 4 – (n + 1)/2^{n-1}$ which tends to 4 as n tends to infinity.

From the point of view of a ring Q of size r, $T(Q) = 2^r – r$. Take for example the hexagonal polyiamond of size 6, which has $2^6 – 6 = 58$ tilings. See A000325.

Theorem 6: An MTP of size >= 6 has a maximum of 2 branch cells; any MTP of size >= 6 is non-treelike

Theorem 7: The core of an MTP (size >= 6) is an MTP

Theorem 8: if P is an MTP with B(P) = 2, then the removal of just one tip of a branch will result in an MTP

Theorem 9: for consecutive integers p,q,r (all >= 6), for at least one of p,q,r there must exist a polyiamond P of that size that is an MTP and has B(P)=0

Theorem 10: if a polyiamond R can be formed from the union of 2 polyiamonds P and Q that touch at just one edge, then T(R) = 2 * T(P) * T(Q)

[3] See definitions in appendix

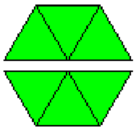
Theorem 11: Consider a polyiamond R that can be formed from the union of 2 polyiamonds P and Q that touch at just two edges of distinct cells. Define the following values (similar to those used in Theorem 5):

- P_A is the number of tilings of P such that the two triangles that border with Q are part of the same tile
- P_B is the number of tilings of P such that the two triangles that border with Q belong to distinct adjacent tiles
- P_C is the number of tilings of P such that the two triangles that border with Q belong to distinct non-adjacent tiles
- Q_A, Q_B and Q_C are as P_A, P_B and P_C respectively.

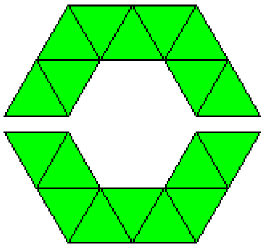
Then $T(R) = 2P_AQ_A + 3P_BQ_A + 4P_CQ_A + 3P_AQ_B + 4P_BQ_B + 4P_CQ_B + 4P_AQ_C + 4P_BQ_C + 4P_CQ_C$

This formula is valid for 2 edges that are both "adjacent" and non-adjacent.

Adjacent:



Non-adjacent:



The formula has 3^2 terms; it can be presumed that a similar formula for polyiamonds touching at 3 edges would have several hundred terms.

Conjectures

It is also possible to make some conjectures:

Conjecture 1: For any given size, n, of polyiamond, the value t, that is the number of tilings of some polyiamond of size n, defines precisely the perimeter and the number of internal, completely surrounded points of all polyiamonds of size n having t tilings.
In other words, for polyiamonds P and Q of the same size, T(P) is equal to T(Q) implies per(P) = per(Q) and int(P) = int(Q).
It should be noted that the opposite is not true. Two polyiamonds of the same size, perimeter and number of internal points may have different numbers of tilings.

Conjecture 2 (stronger, and with even less justification): The value t, that is the number of tilings of some polyiamond of size n, defines precisely the size, the perimeter and the number of internal, completely surrounded points of all polyiamonds having t tilings.
In other words, for polyiamonds P and Q of the same size, T(P) is equal to T(Q) implies size(P) = size(Q), per(P) = per(Q) and int(P) = int(Q).

Conjecture 3: a polyiamond of maximal tilings will have a minimal perimeter for its size.

Conjecture 4 (based on observation (d) above):

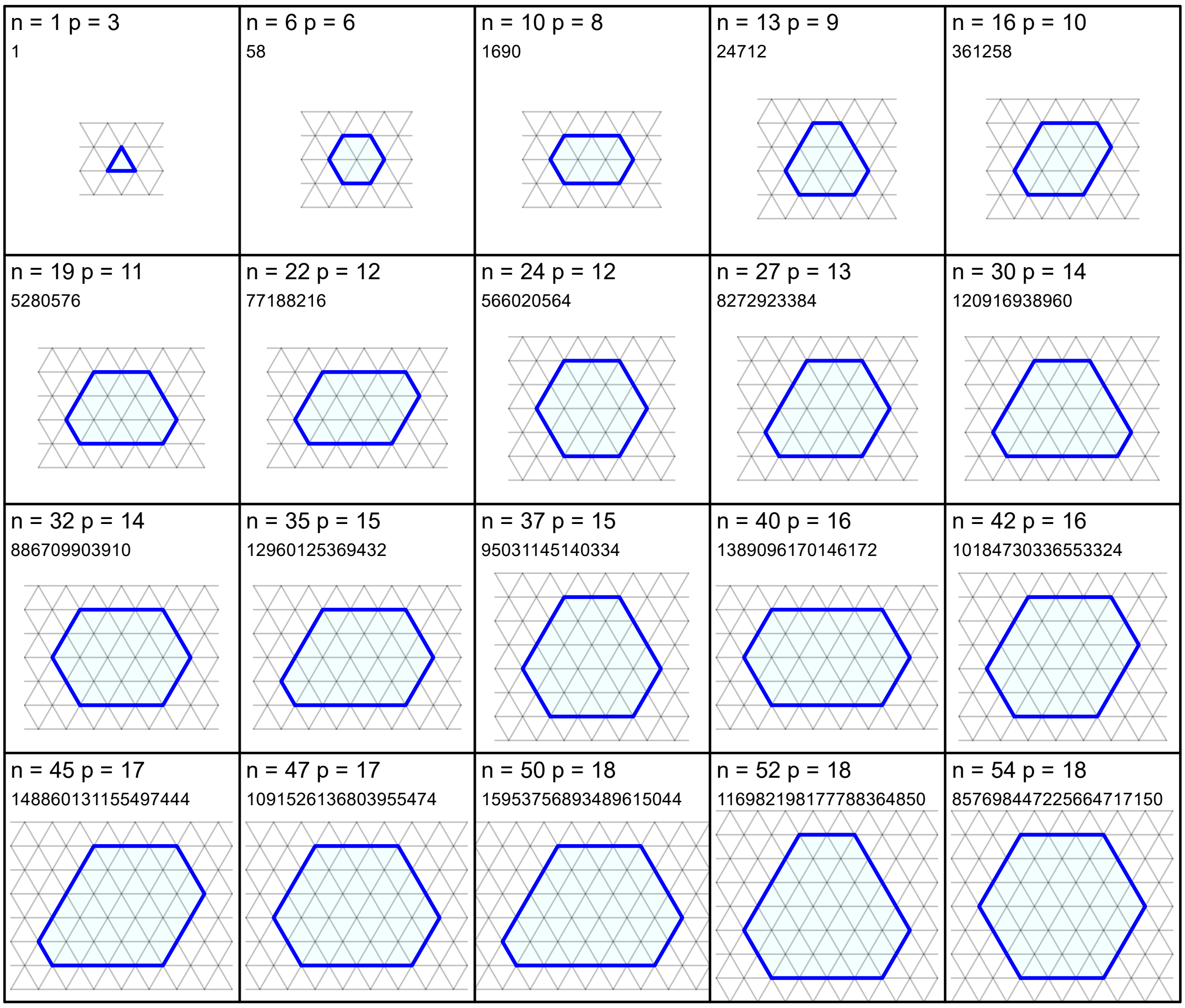
- For polyiamonds P and Q of the same size, per(Q) < per(P) implies T(Q) > T(P).
- For polyiamonds P and Q of the same size and of the same perimeter, int(Q) < int(P) implies T(Q) > T(P)

Conjecture 5: if for some size n there exists a branchless MTP, then there does not exist any MTP of size n with 1 or more branches.

Conjectured maximally tilable polyiamonds.

The following diagram shows those polyiamonds, of various sizes, that have, at the same time minimal perimeter and the maximum number of tilings. Therefore, by applying Conjecture 3 it is possible to extend the table of probable values through to size 54.
The conjecture has been proved correct through to size 22.
For any size i, < 54 but not present in the table, find the highest j < i for which T(j) is known, and then calculate:

$$T(i) = T(j) * 2^{(j-i)}$$



In each case, n gives the size, p the perimeter (minimal), and the number below is the conjectured value of T(n). As already stated, the value is proved for n <= 22.

Conjectured values (proved from size 1 through 22)

n	T(n)
1	1
2	2
3	4
4	8
5	16
6	58
7	116
8	232
9	464
10	1690
11	3380
12	6760
13	24712
14	49424
15	98848
16	361258
17	722516
18	1445032
19	5280576
20	10561152
21	21122304
22	77188216
23	*154376432*
24	*566020564*
25	*1132041128*
26	*2264082256*
27	*8272923384*
28	*16545846768*
29	*33091693536*
30	*120916938960*
31	*241833877920*
32	*886709903910*
33	*1773419807820*
34	*3546839615640*
35	*12960125369432*
36	*25920250738864*
37	*95031145140334*
38	*190062290280668*
39	*380124580561336*
40	*1389096170146172*
41	*2778192340292344*
42	*10184730336553324*
43	*20369460673106648*
44	*40738921346213296*
45	*148860131155497444*
46	*297720262310994888*
47	*1091526136803955474*
48	*2183052273607910948*
49	*4366104547215821896*
50	*15953756893489615044*
51	*31907513786979230088*
52	*116982198177788364850*
53	*233964396355576729700*
54	*857698447225664717150*

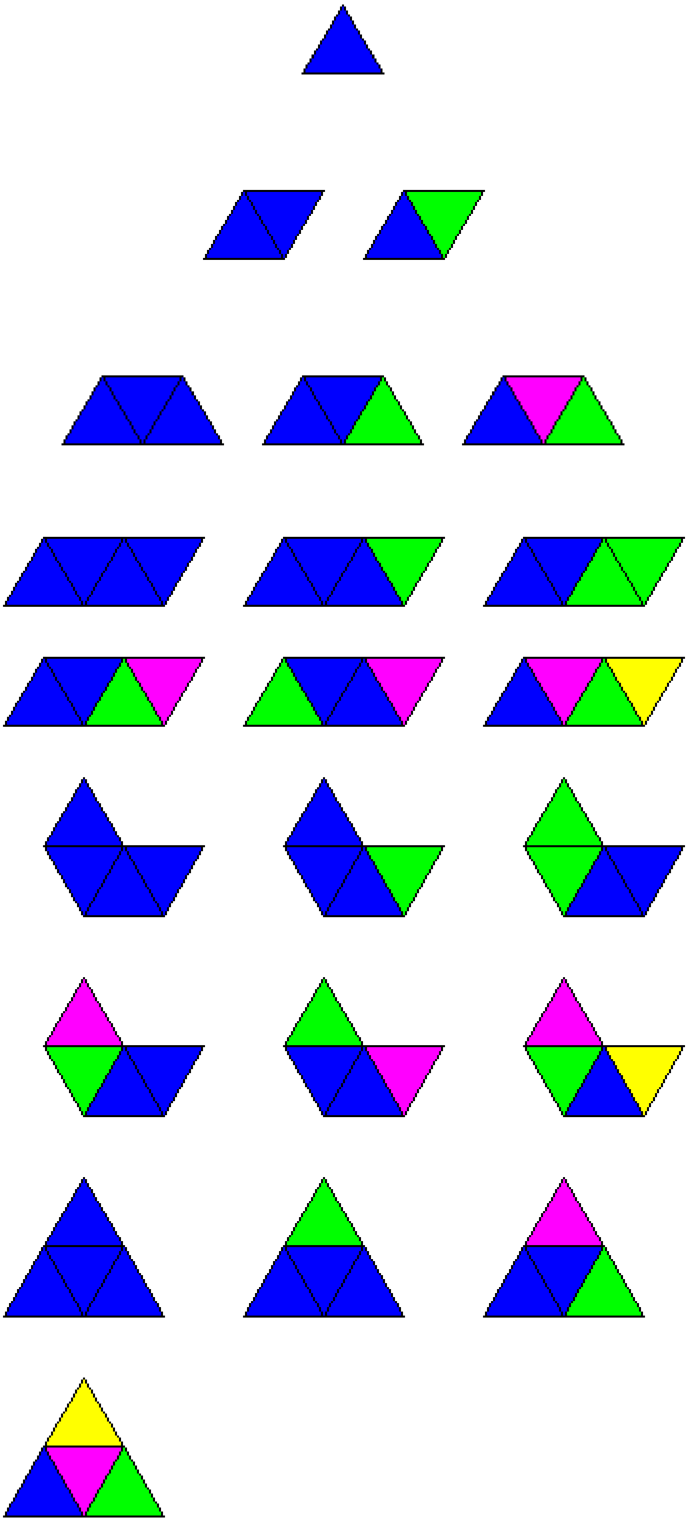
Free tilings

This section discusses the number of tilings a polyiamond may have if reflections and rotations count just once.

The above table has this data for the free case:

1	1(1) 3:0										
2	2(1) 4:0										
3	3(1) 5:0										
4	4(1) 6:0	6(2) 6:0									
5	10(2) 7:0	16(2) 7:0									
6	12(1) 6:1	14(1) 8:0	20(5) 8:0	32(5) 8:0							
7	24(1) 9:0	36(4) 9:0	64(18) 9:0	66(1) 7:1							
8	72(16) 10:0	77(1) 8:1	124(1) 8:1	128(46) 10:0	130(1) 8:1	232(1) 8:1					
9	96(1) 9:1	136(10) 11:0	248(2) 9:1	256(139) 11:0	464(8) 9:1						
10	104(1) 12:0	152(1) 12:0	176(2) 12:0	262(1) 10:1	272(37) 12:0	476(1) 8:2	480(3) 10:1	492(3) 10:1	496(2) 10:1	512(368) 12:0	928(29) 10:1
11	528(27) 13:0	960(9) 11:1	1024(1039) 13:0	1856(109) 11:1	3380(2) 9:2						

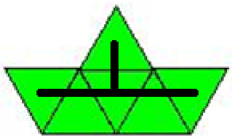
Free tilings look like this:



Appendix

Some definitions.

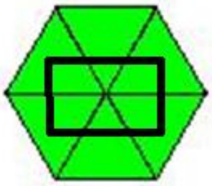
Treelike. A polyiamond is said to be treelike if there is only one path that connects one triangle to another. This diagram shows all the paths of a small treelike polyiamonds:



A treelike polyiamond has a maximal perimeter for its size. In some cases, this specific characteristic is used as the definition treelike.

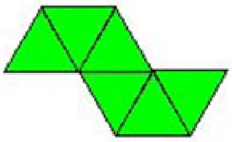
Another definition is that no subset of its cells forms a ring (see definition).

Non-treelike. The opposite of treelike is therefore that there exists at least one pair of triangles in the polyiamond such that there are at least two paths connecting them. In the diagram, it is clear that in this case there are two paths that connect any pair of triangles.

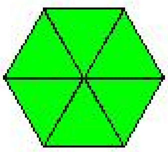
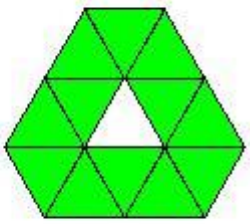


A non-treelike polyiamond has a perimeter that is less than maximal for its size.

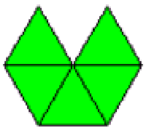
Snake. A polyiamond is a snake if it is treelike and no triangle is adjacent to more than 2 triangles.



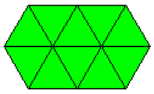
Ring. A polyiamond is a ring if each triangle is adjacent to precisely 2 other triangles.

“Almost-ring” snake: A snake that needs just one more cell to become a ring:



Completely surrounded internal point: A vertex of a triangle of a polyiamond completely surrounded by 6 triangles. In this diagram, we have a polyiamond of size (area) 10, perimeter length 8, and 2 internal points.



Inner edge: an edge of a triangle that is common to 2 adjacent triangles. In the above diagram there are 8 inner edges.

Totally inner edge: an edge of a triangle that is common to 2 adjacent triangles and does not touch any empty space. In the above diagram there is 1 totally inner edge.

Matrix showing how many polyiamonds of each size n have each possible value of size p for the perimeter:

Size	Perimeter																	
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1																	
2		1																
3			1															
4				3														
5					4													
6				1		11												
7					1		23											
8						4		62										
9							11		149									
10						1		38		409								
11							2		118		1066							
12								15		388		2931						
13							1		54		1199		7981					
14								4		233		3763		22166				
15									19		843		11613		61508			
16								1		102		3081		35846		172267		
17									4		454		10620		109941		483088	
18										36		1944		36313		336560		1361475
19									1		190		7669					
20										9		1016		29122				
21											68		4630					
22										3		447		19934				
23											19		2435					
24										1		172		12006				
25											4		1120					
26												54		6499				
27											1		447					
28												16		3126				
29													158					
30												4		1367				
31													47					
32												1		530				
33													13					
34														186				
35													2					
36														56				
37													1					
38														16				
39																		
40														4				
41																		
42														1				
Totals	1	1	1	4	5	16	37	120	345	1181	3844	13429	46736	167172				

Craig Knecht, John Mason & Walter Trump – April 2024