

Proof of Conjecture in A364639
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Notations.

Let $T_{235791}(n, k) = \left\lceil \frac{(n+1)}{k} - \frac{k+1}{2} \right\rceil$ denote the k -th term in the n -th row of the triangle of sequence A235791, $1 \leq n$, $1 \leq k \leq A003056(n)$, and $T_{235791}(n, k) = 0$ for $k > A003056(n)$.

$T_{237591}(n, k) = T_{235791}(n, k) - T_{235791}(n, k+1)$.

$T(n, k) = T_{235791}(n, k) - T_{235791}(n, k+1) - T_{235791}(n-1, k) + T_{235791}(n-1, k+1)$.

$T(k^*(k+1)/2, k) = 1$ for $k \geq 1$.

Claim: $-1 \leq a(n) \leq +1$ holds for every term $a(n)$ in this sequence.

The value of a term in sequence A364639 is determined by some 2×2 square $\begin{pmatrix} s_{i,j} & s_{i,j+1} \\ s_{i+1,j} & s_{i+1,j+1} \end{pmatrix}$ in the triangle of A235791 where the border blanks in positions $(\frac{n}{2}(n+1) - 1, n)$ are replaced by 0's.

Since $0 \leq s_{i+1,j} - s_{i,j} \leq 1$ and $0 \leq s_{i+1,j+1} - s_{i,j+1} \leq 1$ the inequalities

$-1 \leq (s_{i+1,j} - s_{i,j}) - (s_{i+1,j+1} - s_{i,j+1}) \leq +1$ hold.

Claim: Every term $T(n, k)$, $n > A000217(k)$, in column k has period $A002378(k) = k^*(k+1)$.

$T_{235791}(n + k^*(k+1), k) = T_{235791}(n, k) + k + 1$ and $T_{235791}(n + k^*(k+1), k+1) = T_{235791}(n, k) + k$, so that $T(n + k^*(k+1), k) = T_{235791}(n, k) + k + 1 - T_{235791}(n, k) - k - T_{235791}(n-1, k) - k - 1 + T_{235791}(n-1, k) + k = T(n, k)$.

Claim: The sequential periodic sections after the initial term 1 in each column $k \geq 1$ start with $k-1$ 0's and end with k 0's:

After having established periodicity, verifying the claim for the first periodic section is sufficient.

By definition, $T_{235791}(\frac{k}{2}(k+1) + j, k) = 1$ for $1 \leq j < k = A003056(\frac{k}{2}(k+1))$.

For the first set of $k-1$ 0's, $T(\frac{k}{2}(k+1) + j, k) = T_{235791}(\frac{k}{2}(k+1) + j, k) - T_{235791}(\frac{k}{2}(k+1) + j, k+1) - T_{235791}(\frac{k}{2}(k+1) + j - 1, k) + T_{235791}(\frac{k}{2}(k+1) + j - 1, k+1) = 1 - 0 - 1 + 0 = 0$, for $1 \leq j < k$.

For the second set of k 0's, $T(\frac{k}{2}(k+1) + k^2 + j, k) =$

$(T_{235791}(\frac{k}{2}(k+1) + k^2 + j, k) - T_{235791}(\frac{k}{2}(k+1) + k^2 + j, k+1)) - (T_{235791}(\frac{k}{2}(k+1) + k^2 + j - 1, k) - T_{235791}(\frac{k}{2}(k+1) + k^2 + j - 1, k+1)) = ((k+1) - (k-1)) - ((k+1) - (k-1)) = 0$, for $1 \leq j \leq k$.