

# A determinantal formula for the number of trees on $n$ labeled nodes

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May 24, 2023

## Abstract

In this paper we prove that for  $n > 0$  the number of trees on  $n$  labeled nodes is equal to the determinant of  $n C^{-1}(n)$ , where  $C^{-1}(n)$  indicates the inverse of the square Cartan matrix of order  $n - 1$  for the special unitary group of degree  $n$ ,  $SU(n)$ .

## 1 Statement

Given the inverse of the square Cartan matrix of order  $n - 1$ ,  $C^{-1}(n)$ , whose generic element is defined by the following formula [1, 2]

$$C^{-1}(n)_{i,j} = \frac{n \min(i, j) - ij}{n}, \quad (1)$$

it follows that

$$\det[n C^{-1}(n)] = n^{n-2} = A000272(n). \quad (2)$$

which gives the number of trees on  $n$  labeled nodes [3].

## 2 Proof

Let us rewrite the matrix  $n C^{-1}(n)$  as a square array of  $n - 1$  rows and columns

$$n C^{-1}(n) = \begin{bmatrix} n-1 & n-2 & n-3 & \dots & 2 & 1 \\ n-2 & 2(n-2) & 2(n-3) & \dots & 3 & 2 \\ n-3 & 2(n-3) & 3(n-3) & \dots & 4 & 3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 3 & 6 & 9 & \dots & n-4 & n-3 \\ 2 & 4 & 6 & \dots & n-3 & n-2 \\ 1 & 2 & 3 & \dots & n-2 & n-1 \end{bmatrix} \quad (3)$$

and, for each  $j > 1$ , let us subtract  $j$  times the first row from the  $j$ -th row  
[4]

$$\left[ \begin{array}{cccccc} n-1 & n-2 & n-3 & \dots & 2 & 1 \\ -n & 0 & 0 & \dots & 0 & 0 \\ -2n & -n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -(n-4)n & -(n-5)n & -(n-6)n & \dots & 0 & 0 \\ -(n-3)n & -(n-4)n & -(n-5)n & \dots & 0 & 0 \\ -(n-2)n & -(n-3)n & -(n-4)n & \dots & -n & 0 \end{array} \right] \quad (4)$$

If we call the last matrix with  $S(n)$ , using the multilinearity of the determinant and the Laplace expansion along the last column ( $j = n-1$ ) [5], we get

$$\begin{aligned} \det[n C^{-1}(n)] &= \det[S(n)] = (-1)^n (-n)^{n-2} \\ &= (-1)^n (-1)^{n-2} n^{n-2} = n^{n-2} = A000272(n), \end{aligned} \quad (5)$$

which is the statement we need to prove.

QED.

## References

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