The number of musical scales with the natural thirds property

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1 Introduction

The calculation presented here found its origin in a post on Mathstodon by Martin Epstein [1]:

> Most of the musical scales we care about have the "natural thirds" property: if you take any three consecutive notes in the scale (with wraparound) then the distance between the first and last is either 3 or 4 semitones - i.e. a minor or major third. In the 12 tone system there are 33 scales with the natural thirds property: 7 each for the modes of major, melodic minor, harmonic minor, and harmonic major; +1 for whole tone, +2 for symmetric diminished, and +2 for augmented. So $s_{12} = 33$. What can we say about s_n?

We begin by considering a cyclically identified string of n squares $\square\square\cdots\square$. Into these n squares the numbers $0, 1, \ldots, n-1$ will be placed in increasing order. These numbers represent the ordered set of tones from which the scales will be built. We will colour a square black when it is allowed, i.e. when the tone in that square will be included in the scale. A white square represents a discarded tone, i.e. one excluded from the scale. The seven modes of the usual major scale would be represented by

$$\blacksquare\square\blacksquare\square\blacksquare\blacksquare\square\blacksquare\square\blacksquare\square\blacksquare$$

or, equivalently, by any cyclic shift of this pattern.

Notice that we are counting scales by their interval structure, and not by the choice of starting tone. To enforce this we require that every scale begins with the note 0. So, we can place 0 in any of the black squares, then add the remaining tones consecutively, in order, left to right, and with periodic boundary conditions. Those that fall into black squares are in the scale, those in white squares are excluded from the scale. The pattern above contributes 7 to the count because there are seven distinct positions (the black squares) for the 0-tone.

2 Cyclic compositions

We identify three sequences of squares from which all the allowed scales can be built:

$$P_2 = \blacksquare\square, \qquad P_3 = \blacksquare\blacksquare\square, \qquad P_4 = \blacksquare\blacksquare\square\square.$$

For example, the 7 major modes correspond to $P_3P_2P_3P_2P_2$.

The key observation is that P_2 is forbidden from following P_4. This is because $P_4P_2P_j$ will contain a distance of 5 semitones between the three consecutive black squares beginning with the second black square of P_4. The allowed patterns of squares are therefore in one-to-one correspondence with the cyclic compositions of n with parts $2, 3, 4$, but with the subsequence $\ldots, 4, 2, \ldots$ forbidden.

3 Counting distinct scales

If n is prime, then in any given pattern of squares, every black square represents a distinct choice for placing the 0-tone in. In this case we may simply count the total number of black squares and this will be the contribution to s_n. If n is not prime, a pattern of squares might be formed of repeats of a shortest pattern. An example of this in the case $n = 12$ is $P_2P_2P_2P_2P_2P_2$. This pattern has a total of 6 black squares, but represents just one scale, as all six black squares are effectively the same.

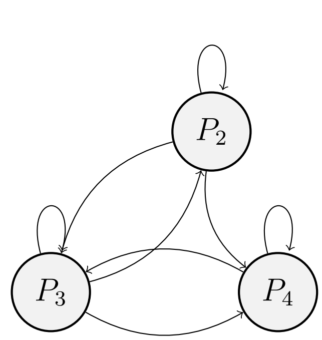
To perform this counting we therefore assign a weight to each of the P_j which is equal to the number of black squares: i.e. $P_2 \to 1$, $P_3 \to 2$, and $P_4 \to 2$. These weights are then summed for each pattern of squares, but if a pattern is formed of q repeats of a (shortest) sub-pattern, the sum must be divided by q.

The sequence (A361378 in the OEIS) can be computed using this algorithm (beginning with $n = 1$):

$$s_n = 0, 1, 2, 3, 3, 3, 8, 8, 12, 16, 25, 33, 45, 66, 91, 128, 177, 252, 351, \ldots$$

4 Transfer matrix method

We will consider all walks of k steps on the following directed graph,



with the restriction that if we begin on P_2 we must not end on P_4. This last restriction prevents the $4, 2$ subsequence in the wrap-around case. We aim to form a generating function with the exponent of t counting n, the exponent of w counting the number of black squares, and the exponent of z counting the number k of steps (i.e. the number of parts in the composition). We would therefore like to multiply by wt^2 for each visit to P_2, w^2t^3 for each visit to P_3 and w^2t^4 for each visit to P_4. The transfer matrix is therefore

$$M = \begin{pmatrix} wt^2 & wt^2 & 0 \\ w^2t^3 & w^2t^3 & w^2t^3 \\ w^2t^4 & w^2t^4 & w^2t^4 \end{pmatrix}$$

for generic transitions, and

$$M_0 = \begin{pmatrix} wt^2 & wt^2 & 0 \\ w^2t^3 & w^2t^3 & w^2t^3 \\ 0 & 0 & 0 \end{pmatrix}$$

for the last transition in the case that we started on P_2.

It turns out that counting the black squares of the patterns produced by the transfer matrix method, then dividing by the number k of parts, performs our intended enumeration. This is because if the pattern is not composed of repeats of a sub-pattern, the transfer matrix method will count it k times, once for each cyclic ordering of the parts. If it is composed of q repeats of a sequence of p parts $P_{j_1} \cdots P_{j_p}$, and so $k = qp$, the transfer matrix method will count it p times (once for each choice of starting part). Hence, after dividing the contribution by k, the corresponding cyclic composition's contribution is multiplied by $p/k = 1/q$, as required.

5 The generating function

Let $\mathbf{e} = (1, 1, 1)$ and

$$\mathbf{u} = \begin{pmatrix} wt^2 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 0 \\ w^2t^3 \\ w^2t^4 \end{pmatrix}.$$

The generating function is then formed as follows. Let

$$F(z, w, t) = zwt^2 + \sum_{k\geq 2} z^k\, \mathbf{e} M_0 M^{k-2} \mathbf{u} + \sum_{k\geq 1} z^k\, \mathbf{e} M^{k-1} \mathbf{v}.$$

This produces all the walks of interest on the directed graph. The first term represents the pattern P_2 which is only relevant for the case $n = 2$. The second term counts the patterns of k parts which begin with P_2 (and hence must not end with P_4). The last term counts those patterns which begin with either P_3 or P_4.

The infinite sums are geometric and hence we have

$$F(z, w, t) = zwt^2 + z^2\, \mathbf{e} M_0 (\mathbb{I} - zM)^{-1} \mathbf{u} + z\, \mathbf{e} (\mathbb{I} - zM)^{-1} \mathbf{v},$$

which is easily calculated

$$F(z, w, t) = \frac{t^2wz(1 + tw(1 + t - 2t^3wz))}{1 + t^2wz(-1 + tw(-1 - t + t^3wz))}.$$

In order to find the generating function for s_n we define $s(t) = \sum s_n t^n$, and compute

$$s(t) = \mathcal{F}(1, 1, t),$$

where

$$\mathcal{F}(z, w, t) = \int_0^z \frac{d\zeta}{\zeta} \frac{\partial}{\partial w} F(\zeta, w, t).$$

The derivative in w sums the weights (i.e. the black squares), whilst the integral divides the sum of weights by the number k of parts as required. The result is

$$s(t) = \frac{t^2 + 2t^3 + 2t^4 - 3t^6}{1 - t^2 - t^3 - t^4 + t^6}.$$

6 Recursion relation

The generating function suggests the recursion relation for $n > 6$

$$s_n = s_{n-2} + s_{n-3} + s_{n-4} - s_{n-6}.$$

It is relatively straightforward to understand this relation: to count s_n we can consider all s_{n-2} scales built from two less tones and complete them by placing the two extra tones into a P_2 placed at the end, all s_{n-3} scales built on three less tones completed similarly with P_3, and all s_{n-4} scales on four less tones completed with P_4. In so doing we will introduce the forbidden subsequence $P_4 P_2$ and these scales must be removed; this is accomplished by subtracting the s_{n-6} scales built from six less tones completed with $P_4 P_2$.

References

[1] Epstein, Martin, https://mathstodon.xyz/@rivfader/109916586537191508, February 23, 2023.