Tilings of a 2X2Xn box with 3d-tiles

I) Introduction

These tiles will be considered: 1X1X1-cubes, 1X1X2-cuboids (dominos), 2X2X1-cuboids (plates) and trominos (L-shaped combination of three 1X1X1-cubes).
The aim is to find the number a(n) of tilings for any combination of tiles. Let us assume that the box (standing upright) is tiled bottom-up, using certain blocks.
Definition: A block is a combination of tiles with j=1,2,3,4 lower cubes and k=0,1,2,3,4 upper cubes. Restriction: if j=4 then k=0.
In the following chapter, all transitions from one level to the next one (or the following next one in some cases), using blocks, will be analyzed. Each block has a binary code $1, \dots, 13$:

$$Single\ cube \rightarrow 1, \quad domino \rightarrow 2, \quad tromino \rightarrow 4, \quad plate \rightarrow 8.$$

Example: A block of single cubes and trominos has the code $5 = 1 + 4$.
Codes 14 and 15 do not occur because a block contains at most 8 cubes.
After finding all transitions, recursive formulas for a(n) can be derived.

II) Transitions using blocks

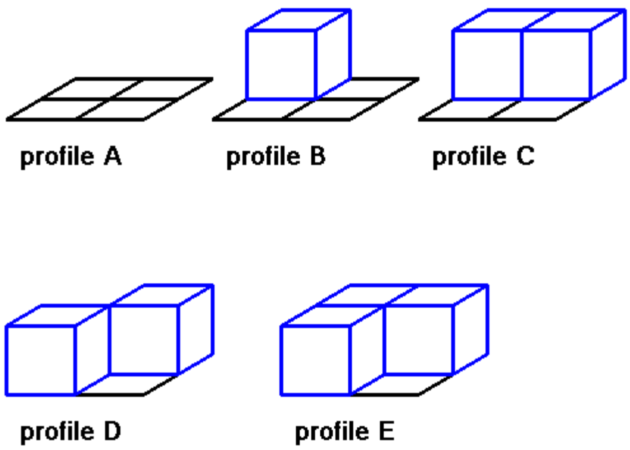

Fig. 1

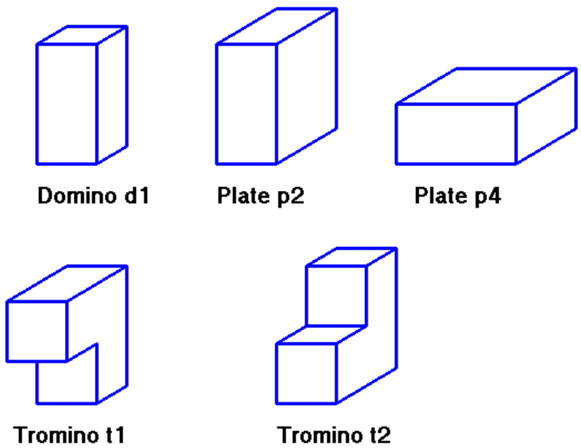

Fig. 2

Fig. 1 shows five different profiles belonging to the same level such that j cubes standing (alone or as a part of another tile) on profile A: $A(j = 0), B(j = 1), C\ and\ D(j = 2), E(j = 3)$.
There are four rotation images of the profiles B, C, E and two images of profile D.
Moreover, the orientation of a tile must be considered: $c1, d1, d2, t1, t2, t3, p4$. The letter stands for “single cube”, “domino”, “tromino” or “plate” and the index for the number of cubes touching the current ground. If the cubes on profile C and E are connected, they represent d2 and t3, respectively.

The profiles have $4 - j$ empty squares ("holes"). Example: By filling the hole on profile E with a single cube, one generates profile A on the next level. Notation: c1: E0->1A1.
E0 is the E-profile on the current level, A1 is the A-profile on the next level. "1A1" means that there is only one way of generating A1. In 2d2: A0 -> 2A1, there are two ways of placing the dominos and in 4d1: A0 -> 1A2, the following next A-profile is generated.
It is useful to make a difference between flat tiles (c_1, d_2, t_3, p_4) and upright tiles (d_1, t_1, t_2, p_2).

All blocks (and the depending transitions) can be found in this order:

(1) Blocks without single cubes
- a) only upright tiles
- b) both tiles (mixed tiling)
- c) only flat tiles

(2) Blocks with single cubes
- a) single cubes and other tiles (mixed tiling)
- b) only single cubes

Job (1a)
For a systematic search, the tiles are written as vectors:

$$d1 = \binom{2}{1}, t1 = \binom{3}{1}, t2 = \binom{3}{2}, p2 = \binom{4}{2}$$

The upper component is the number of all cubes, the lower one is the number of cubes touching the current ground. Any block can be written as

$$p * \binom{2}{1} + q * \binom{3}{1} + r * \binom{3}{2} + s * \binom{4}{2} = \binom{x}{y}$$

with $p, q, r, s \geq 0$ and $0 < y \leq 4$ and $0 < x \leq 8$ and $0 \leq x - y \leq 4$.
y is the number of lower cubes, $x - y$ is the number of upper cubes such that
$y = 1,2,3,4$ corresponds to the profiles $E0, D0$ or $C0, B0, A0$ and
$x - y = 1,2,3,4$ corresponds to the profiles $B1, C1$ or $D1, E1, A2$.
Note that four upper cubes generate the following next profile $A2$.

There are 20 quadruples (p, q, r, s) solving the equation:

$pqrs$	xy	Block	Code	Transition with multiplicity
1 0 0 0	2 1	d1	2	E0->1*B1
2 0 0 0	4 2	2d1	2	C0->1*C1, D0->1*D1
3 0 0 0	6 3	3d1	2	B0->1*E1
4 0 0 0	8 4	4d1	2	A0->1*A2
0 1 0 0	3 1	t1	4	E0->2*C1
1 1 0 0	5 2	d1 +t1	6	C0->2*E1, D0->2*E1
2 1 0 0	7 3	2d1 +t1	6	B0->2*A2
0 2 0 0	6 2	2t1	4	C0->2*A2, D0->2*A2
0 0 1 0	3 2	t2	4	C0->2*B1
1 0 1 0	5 3	d1 +t2	6	B0->2*C1 + 2*D1
2 0 1 0	7 4	2d1 +t2	6	A0->8*E1

0 1 1 0	6 3	t1 +t2	4	B0->4*E1
1 1 1 0	8 4	d1 +t1 +t2	6	A0->8*A2
0 0 2 0	6 4	2t2	4	A0->4*C1 + 4*D1
0 0 0 1	4 2	p2	8	C0->2*C1
1 0 0 1	6 3	d1 +p2	10	B0->2*E1
2 0 0 1	8 4	2d1 +p2	10	A0->4*A2
0 1 0 1	7 3	t1 +p2	12	B0->2*A2
0 0 1 1	7 4	t2 +p2	12	A0->8*E1
0 0 0 2	8 4	2p2	8	A0->2*A2

Table 1a

Job (1b)

Ignoring single cubes, exactly one flat tile can be used in a mixed tiling. A recursion can be based on this fact. Example: A0 + t3 is equivalent to profile E0.

Notation: A0 + t3 => 4E0 (there are four ways of generating E0).

Using the two tilings d1: E0 -> 1B1 and t1: E0 -> 2C1 (see table above), one obtains

t3 + d1, A0 -> 4B1 and t3 + t1, A0 -> 8C1.

This way, two mixed tilings are derived from one "macro tiling" A0 + t3 => 4E0.

Tiles	Code	**Macro** / Transition	Block	Code	**Macro** / Transition
		A0 + d2 => 4C0			**A0 + t3 => 4E0**
d2 + 2d1	2	A0 -> 4C1	t3 + d1	6	A0 -> 4B1
d2 + 2t1	6	A0 -> 4A2	t3 + t1	4	A0 -> 8C1
d2 + d1 + t1	6	A0 -> 8E1			**B0 + d2 => 2E0**
d2 + p2	10	A0 -> 4C1	d2 + d1	2	B0 -> 2B1
d2 + t2	6	A0 -> 8B1	d2 + t1	6	B0 -> 4C1

Table 1b

Job (1c)

Block	Code	Transition
d2	2	C0->1*A1
2d2	2	A0->2*A1
t3	4	B0->1*A1
p4	8	A0->1*A1

Table 1c

Annotation:

If job (1c) is done before (1b), double counting can occur. Example:
A0 + d2 => 4C0 and d2: C0->1*A1 leads to 2d2: A0->4*A1 instead of 2d2: A0->2*A1. For the same reason, job (2a) must be done before (2b).

Job (2a)

All mixed tilings using c1 can be derived from macro tilings:

Tiles	Code	**Macro** / Transition	Block	Code	**Macro** / Transition
		A0 + c1 => 4B0			**A0 +3c1 => 4E0**
c1 + 2d1 + t1	7	A0 -> 8A2	3c1 + d1	3	A0 -> 4B1
c1 + 3d1	3	A0 -> 4E1	3c1 + t1	5	A0 -> 8C1
c1 + d1 + t2	7	A0 -> 8C1			**B0 + c1 => 2C0**
c1 + d1 + t2	7	A0 -> 8D1	c1 + 2d1	3	B0 -> 2C1
c1 + p2 + d1	11	A0 -> 8E1	c1 + 2t1	5	B0 -> 2A2
c1 + p2 + t1	13	A0 -> 8A2	c1 + d1 + t1	7	B0 -> 4E1
c1 + t1 + t2	5	A0 -> 16E1	c1 + p2	9	B0 -> 2C1
c1 + d2 + d1	3	A0 -> 8B1	c1 + t2	5	B0 -> 4B1
c1 + d2 + t1	7	A0 -> 16C1	c1 + d2	3	B0 -> 2A1
c1 + t3	5	A0 -> 4A1			**B0 + c1 => 1D0**
2c1		**A0 => 4C0**	c1 + 2d1	3	B0 -> 1D1
2c1 + 2d1	3	A0 -> 4C1	c1 + 2t1	5	B0 -> 2A2
2c1 + 2t1	5	A0 -> 4A2	c1 + d1 + t1	7	B0 -> 2E1
2c1 + d1 + t1	7	A0 -> 8E1			**B0 + 2c1 => 3E0**
2c1 + p2	9	A0 -> 4C1	2c1 + d1	3	B0 -> 3B1
2c1 + t2	5	A0 -> 8B1	2c1 + t1	5	B0 -> 6C1
2c1 + d2	3	A0 -> 4A1			**C0 + c1 => 2E0**
		A0 + 2c1 => 2D0	c1 + d1	3	C0 -> 2B1
2c1 + 2d1	3	A0 -> 2D1	c1 + t1	5	C0 -> 4C1
2c1 + 2t1	5	A0 -> 4A2			**D0 + c1 => 2E0**
2c1 + d1 + t1	7	A0 -> 4E1	c1 + d1	3	D0 -> 2B1
			c1 + t1	5	D0 -> 4C1

Table 2a

Job (2b)

Block	Code	Transition	Block	Code	Transition
4c1	1	A0 -> 1A1	2c1	1	D0 -> 1A1
3c1	1	B0 -> 1A1	c1	1	E0 -> 1A1
2c1	1	C0 -> 1A1			

Table 2b

Example: Transitions, only using trominos

Compact encoding (used in a Maxima program), example:
There are two transitions with code 12: A0->8*E1 and B0->2*A2
New representation as triples: [[1,5,8],[2,6,2]].
First component: 1,2,3,4 or 5 standing for A0,B0,C0,D0 or E0
Second component: 1,2,3,4,5 or 6 standing for A1,B1,C1,D1,E1 or A2
The third component stands for the multiplicity of the transition.

Complete overview:

Code	Transitions
1	[[1,1,1],[2,1,1],[3,1,1],[4,1,1],[5,1,1]]
2	[[1,1,2],[1,3,4],[1,6,1],[2,2,2],[2,5,1],[3,1,1],[3,3,1],[4,4,1],[5,2,1]]
3	[[1,1,4],[1,2,12],[1,3,4],[1,4,2],[1,5,4],[2,1,2],[2,2,3],[2,3,2],[2,4,1],[3,2,2],[4,2,2]]
4	[[1,3,12],[1,4,4],[2,1,1],[2,5,4],[3,2,2],[3,6,1],[4,6,2],[5,3,2]]
5	[[1,1,4],[1,2,8],[1,3,8],[1,5,16],[1,6,8],[2,2,4],[2,3,6],[2,6,4],[3,3,4],[4,3,4]]
6	[[1,2,12],[1,5,16],[1,6,12],[2,3,6],[2,4,2],[2,6,2],[3,5,2],[4,5,2]]
7	[[1,3,24],[1,4,8],[1,5,12],[1,6,8],[2,5,6]]
8	[[1,1,1],[1,6,2],[3,3,1]]
9	[[1,3,4],[2,3,2]]
10	[[1,3,4],[1,6,4],[2,5,2]]
11	[[1,5,8]]
12	[[1,5,8],[2,6,2]]
13	[[1,6,8]]

Each line represents a matrix $L(code, i, k)$,
Example: $L(12,1,5) = 8, L(12,2,6) = 2$ and $L(12, i, k) = 0$ otherwise.

III) Derivation of recursive formulas

The matrices $M(code) = L^T(code)$ (transposed matrix) will be used to derive all recursive formulas. Selecting a combination of tiles, we have to make a difference between a local tiling described by the code of a block and the global tiling for the whole box described by the type, which is a binary number, too. Example: Type 5 means that only single cubes and trominos are allowed, i.e. blocks with codes 1,4,5. The set of codes y belonging to a type x can be described by the function "bitand": $\{y|y\ bitand\ x = y\}$.

Example 1 (Only dominos are used, type 2, code 2)

Table 3a shows the matrix $L(2)$. For level n and for each profile, let the number of tilings be a(n), b(n), c(n), d(n) and e(n) respectively. By transposing $L(2)$, one obtains $M(2)$ in table 3b as a recurrence (j=2 for the last row and j=1 otherwise):

	A1	B1	C1	C1	E1	A2
A0	2	0	4	0	0	1
B0	0	2	0	0	1	0
C0	1	0	1	0	0	0
D0	0	0	0	1	0	0
E0	0	1	0	0	0	0

Table 3a

	a(n-j)	b(n-j)	c(n-j)	d(n-j)	e(n-j)
a(n)	2	0	1	0	0
b(n)	0	2	0	0	1
c(n)	4	0	1	0	0
d(n)	0	0	0	1	0
e(n)	0	1	0	0	0
a(n)	1	0	0	0	0

Table 3b

Table 3b more extensively:

$$\begin{pmatrix} a(n) \\ b(n) \\ c(n) \\ d(n) \\ e(n) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} a(n-1) \\ b(n-1) \\ c(n-1) \\ d(n-1) \\ e(n-1) \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} a(n-2) \\ b(n-2) \\ c(n-2) \\ d(n-2) \\ e(n-2) \end{pmatrix}$$

Recurrence:
a(n) = 2*a(n-1) + c(n-1) + a(n-2), b(n) = 2*b(n-1) + e(n-1),
c(n) = 4*a(n-1) + c(n-1), d(n) = d(n-1), e(n) = b(n-1)
with a(n)= b(n)=c(n)=d(n)=e(n)=0 for n≤0, except for a(0)=1.
Simplification: a(n) = 2*a(n-1) + c(n-1) + a(n-2), c(n) = 4*a(n-1) + c(n-1)
Result: a(n) = 1, 2, 9, 32, 121, 450, 1681, 6272, 23409, 87362, 326041, .. = A006253(n)

Example 2 (Only plates are used, type 8, code 8)

$M(8)$ leads to a(n) = a(n-1) + 2*a(n-2),

a(n)= 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683,… = A001045(n)

Example 3 (Plates and single cubes, type 9, codes 1,8,9)

The matrix corresponding to 3b now is $M(1) + M(8) + M(9)$:

2	1	1	1	1	Recurrence:
0	0	0	0	0	a(n) = 2*a(n-1) + c(n-1) + 2*a(n-2)
4	2	1	0	0	c(n) = 4*a(n-1) + c(n-1)
0	0	0	0	0	
0	0	0	0	0	
2	0	0	0	0	

Result: a(n) = 1, 2, 10, 36, 144, 556, 2172, 8452, 32932, 128260, … = A335559(n)

Example 4 (Plates and dominos, type 10, codes 2,8,10)

Matrix $M(2) + M(8) + M(10)$:

3	0	1	0	0	Recurrence:
0	2	0	0	1	a(n) = 3*a(n-1) + c(n-1) + 7*a(n-2), b(n) = 2*b(n-1) + e(n-1)
8	0	2	0	0	c(n) = 8*a(n-1) + 2*c(n-1), d(n) = d(n-1), e(n) = 3*b(n-1)
0	0	0	1	0	Obviously: b(n)=d(n)=e(n)=0 for n≥0
0	3	0	0	0	
7	0	0	0	0	

Simplification: a(n) = 3*a(n-1) + c(n-1) + 7*a(n-2), c(n) = 8*a(n-1) + 2*c(n-1)

Result: a(n) = 1, 3, 24, 133, 839, 5056, 30969, 188603, 1150952, 7018621, …

Example 5 (Trominos, type 4, code 4)

Matrix $M(4)$:

0	1	0	0	0	Recurrence:
0	0	2	0	0	a(n) = b(n-1) + c(n-2) + 2*d(n-2), b(n) = 2*c(n-1)
12	0	0	0	2	c(n) = 12*a(n-1) + 2*e(n-1), d(n) = 4*a(n-1), e(n) = 4*b(n-1)
4	0	0	0	0	
0	4	0	0	0	
0	0	1	2	0	a(n) = a(n) = 1, 0, 0, 44, 0, 0, 2512, 0, 0, 145088, …

Simplification: a(n) = 44*a(n-3) + 6*e(n-3), e(n) = 96*a(n-3) +16*e(n-3)

With the transformation a(3*n) -> a(n), e(3*n) -> e(n), one obtains:

a(n) = 44*a(n-1) + 6*e(n-1), e(n) = 96*a(n-1) +16*e(n-1) and

a(n) = 1, 44, 2512, 145088, 8383744, 484453376, …

Example 6 (Plates and trominos, type 12, codes 4,8,12)

Matrix $M(4) + M(8) + M(12)$:

1	1	0	0	0	b(n)=d(n)=e(n)=0 for n≥0 Recurrence: a(n) = a(n-1) + b(n-1) + 2*a(n-2) + 2*b(n-2) + c(n-2) + 2*d(n-2), b(n) = 2*c(n-1), c(n) = 12*a(n-1) + c(n-1) + 2*e(n-1), d(n) = 4*a(n-1), e(n) = 8*a(n-1) + 4*b(n-1)
0	0	2	0	0	
12	0	1	0	2	
4	0	0	0	0	
8	4	0	0	0	
2	2	1	2	0	

Simplification: a(n) = a(n-1) + 3*c(n-2) + 2*a(n-2) + 4*c(n-3) + 8*a(n-3)

c(n) = 12*a(n-1) + c(n-1) + 16*a(n-2) + 16*c(n-3)

Result: a(n) = 1, 1, 3, 49, 231, 789, 4771, 27225, 122799, 607469, 3255979, …

Example 7 (Dominos and single cubes, type 3, codes 1,2,3)

Matrix $M(1) + M(2) + M(3)$:

7	3	2	1	1	Recurrence: a(n) = 7*a(n-1) + 3*b(n-1) + 2*c(n-1) + d(n-1) + e(n-1) + a(n-2) b(n) = 12*a(n-1) + 5*b(n-1) + 2*c(n-1) + 2*d(n-1) + e(n-1) c(n) = 8*a(n-1) + 2*b(n-1) + c(n-1), d(n) = 2*a(n-1) + b(n-1) +d(n-1)) e(n) = 4*a(n-1) + b(n-1)
12	5	2	2	1	
8	2	1	0	0	
2	1	0	1	0	
4	1	0	0	0	
1	0	0	0	0	

a(n) = 1, 7, 108, 1511, 21497, 305184, 4334009, 61545775, 873996300, 12411393231, …

Example 8 (Trominos and single cubes, type 5, codes 1,4,5)

Matrix $M(1) + M(4) + M(5)$:

5	2	1	1	1	Recurrence: a(n) = 5*a(n-1) + 2*b(n-1) + c(n-1) + d(n-1) + e(n-1) + 8*a(n-2) + 4*b(n-2) + c(n-2) + 2*d(n-2) b(n) = 8*a(n-1) + 4*b(n-1) + 2*c(n-1) c(n) = 20*a(n-1) + 6*b(n-1) + 4*c(n-1) + 4*d(n-1) + 2*e(n-1) d(n) = 4*a(n-1), e(n) = 16*a(n-1) + 4*b(n-1)
8	4	2	0	0	
20	6	4	4	2	
4	0	0	0	0	
16	4	0	0	0	
8	4	1	2	0	

a(n) = 1, 5, 89, 1177, 16873, 237977, 3366793, 47599097, 673035625, 9516252633, …

Example 9 (Trominos and dominos, type 6, codes 2,4,6)

Matrix $M(2) + M(4) + M(6)$:

2	1	1	0	0	Recurrence: a(n) = 2*a(n-1) + b(n-1) + c(n-1) + 13*a(n-2) + 2*b(n-2) + c(n-2) + 2*d(n-2) b(n) = 12*a(n-1) + 2*b(n-1) + 2*c(n-1) + e(n-1) c(n) = 16*a(n-1) + 6*b(n-1) + c(n-1) + 2*e(n-1) d(n) = 4*a(n-1) + 2*b(n-1) + d(n-1) e(n) = 16*a(n-1) + 5*b(n-1) + 2*c(n-1) + 2*d(n-1)
12	2	2	0	1	
16	6	1	0	2	
4	2	0	1	0	
16	5	2	2	0	
13	2	1	2	0	

a(n) = 1, 2, 45, 412, 4705, 50374, 549109, 5955544, 64683649, 702259786, 7625147293,…

Example 10 (Trominos, dominos and single cubes, type 7, codes 1,2,3,4,5,6,7)

Matrix $M(1) + M(2) + M(3) + M(4) + M(5) + M(6) + M(7)$:

11	4	2	1	1	Recurrence:
32	9	4	2	1	a(n) = 11*a(n-1) + 4*b(n-1) + 2*c(n-1) + d(n-1) + e(n-1) + 29*a(n-2) + 6*b(n-2) + c(n-2) + 2*d(n-2)
52	14	5	4	2	b(n) = 32*a(n-1) + 9*b(n-1) + 4*c(n-1) + 2*d(n-1) + e(n-1)
14	3	0	1	0	c(n) = 52*a(n-1) + 14*b(n-1) + 5*c(n-1) + 4*d(n-1) + 2*e(n-1)
48	11	2	2	0	d(n) = 14*a(n-1) + 3*b(n-1) + d(n-1)
29	6	1	2	0	e(n) = 48*a(n-1) + 11*b(n-1) + 2*c(n-1) + 2*d(n-1)

a(n) = 1, 11, 444, 13311, 422617, 13265660, 417336617, 13123557903, 412719195520, …

Example 11 (Plates, dominos and single cubes, type 11, codes 1,2,3,8,9,10,11)

Matrix $M(1) + M(2) + M(3) + M(8) + M(9) + M(10) + M(11)$:

8	3	2	1	1	Recurrence:
12	5	2	2	1	a(n) = 8*a(n-1) + 3*b(n-1) + 2*c(n-1) + d(n-1) + e(n-1) + 7*a(n-2)
16	4	2	0	0	b(n) = 12*a(n-1) + 5*b(n-1) + 2*c(n-1) + 2*d(n-1) + e(n-1)
2	1	0	1	0	c(n) = 16*a(n-1) + 4*b(n-1) + 2*c(n-1)
12	3	0	0	0	d(n) = 2*a(n-1) + b(n-1) + d(n-1)
7	0	0	0	0	e(n) = 12*a(n-1) + 3*b(n-1)

a(n) = 1, 8, 153, 2470, 41571, 693850, 11602579, 193942076, 3242104149, 54196828452, …

Example 12 (Plates, trominos and single cubes, type 13, codes 1, 4, 5, 8, 9, 12, 13)

Matrix $M(1) + M(4) + M(5) + M(8) + M(9) + M(12) + M(13)$:

6	2	1	1	1	Recurrence:
8	4	2	0	0	a(n) = 6*a(n-1) + 2*b(n-1) + c(n-1) + d(n-1) + e(n-1) + 18*a(n-2) + 6*b(n-2) + c(n-2) + 2*d(n-2)
24	8	5	4	2	b(n) = 8*a(n-1) + 4*b(n-1) + 2*c(n-1)
4	0	0	0	0	c(n) = 24*a(n-1) + 8*b(n-1) + 5*c(n-1) + 4*d(n-1) + 2*e(n-1)
24	4	0	0	0	d(n) = 4*a(n-1)
18	6	1	2	0	e(n) = 24*a(n-1) + 4*b(n-1)

a(n) = 1, 6, 122, 1768, 28844, 457592, 7318760, 116806896, 1865305376, 29782666544, …

Example 13 (Plates, trominos and dominos, type 14, codes 2, 4, 6, 8, 10, 12)

Matrix $M(2) + M(4) + M(6) + M(8) + M(10) + M(12)$:

3	1	1	0	0	Recurrence:
12	2	2	0	1	a(n) = 3*a(n-1) + b(n-1) + c(n-1) + 19*a(n-2) + 4*b(n-2) + c(n-2) + 2*d(n-2)
20	6	2	0	2	b(n) = 12*a(n-1) + 2*b(n-1) + 2*c(n-1) + e(n-1)
4	2	0	1	0	c(n) = 20*a(n-1) + 6*b(n-1) + 2*c(n-1) + 2*e(n-1)
24	7	2	2	0	d(n) = 4*a(n-1) + 2*b(n-1) + d(n-1)
19	4	1	2	0	e(n) = 24*a(n-1) + 7*b(n-1) + 2*c(n-1) + 2*d(n-1)

a(n) = 1, 3, 60, 657, 8311, 101284, 1246049, 15292819, 187803572, 2305968393, …

Example 14 (Plates, trominos, dominos and single cubes, type 15, all codes 1,…,13)

Matrix $M(1) + \cdots + M(13)$:

12	4	2	1	1
32	9	4	2	1
60	16	6	4	2
14	3	0	1	0
64	13	2	2	0
43	8	1	2	0

Recurrence:

a(n) = 12*a(n-1) + 4*b(n-1) + 2*c(n-1) + d(n-1) + e(n-1)
    + 43*a(n-2) + 8*b(n-2) + c(n-2) + 2*d(n-2)

b(n) = 32*a(n-1) + 9*b(n-1) + 4*c(n-1) + 2*d(n-1) + e(n-1)

c(n) = 60*a(n-1) + 16*b(n-1) + 6*c(n-1) + 4*d(n-1) + 2*e(n-1)

d(n) = 14*a(n-1) + 3*b(n-1) + d(n-1)

e(n) = 64*a(n-1) + 13*b(n-1) + 2*c(n-1) + 2*d(n-1)

a(n) = 1, 12, 513, 16194, 547543, 18234354, 609298887, 20344385080, 679408772089, …