A proof that A359506 is a bijection from the nonnegative integers to the nonpowers of 2.

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Definition. Say that a finite sequence of nonnegative integers $\{b_i\}_{i=1}^t$ has the **zero XOR property** if

$$b_1 \oplus b_2 \oplus \cdots \oplus b_t = 0.$$

Definition. Let $f(n)$ be the least nonnegative integer k such that there exists an increasing sequence

$$k = b_1 < b_2 < \cdots < b_t = n$$

with the zero XOR property.

Lemma. If n is not a power of 2, then $f(n)$ is well-defined.

Proof. It is enough to show that the set of increasing sequences of nonnegative integers with the zero XOR property ending with n is nonempty. If n is not a power of 2, then one can use the sequence where $b_t = n$ and $b_1, b_2, \ldots, b_{t-1}$ is given by the n-th row of A348296. □

Lemma. If n is a power of 2, then $f(n)$ is not well-defined.

Proof. No increasing sequence of nonnegative integers that end with a power of two can have the zero XOR property because all integers less than n do not have a long enough binary representation to remove the leading 1 in the binary representation of n. □

Theorem 1. A359506 *is surjective onto the non-powers of* 2*.*

Proof. It is enough to show that $f(\text{A359506}(n)) = n$.

First, notice that $f(\text{A359506}(n)) \geq n$ because we know from A359506 that there exists a sequence starting with n and ending with A359506(n). Then, for sake of contradiction, assume that $f(\text{A359506}(n)) = m > n$. This means that there exists a sequence

$$m = c_1 < c_2 < \cdots < c_s = \text{A359506}(n)$$

with the zero XOR property. By the definition of A359506, there exists a sequence

$$n = b_1 < b_2 < \cdots < b_t = \text{A359506}(n).$$

Because $n < m$, if we take the symmetric difference of these two sequences we end up with a sequence with the zero XOR property that begins with n and ends with a value less than A359506(n), a contradiction of the definition of A359506. Therefore, f is a left inverse for A359506, and A359506 is a bijection from the nonnegative integers to the nonpowers of 2. □