Proof of the conjectures stated in A359087

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Sequence A359087 is defined as follows. For each $n \in \mathbb{N}$ we construct a reversed pyramid whose top row is

$$1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1,$$

and in which each element below is the sum of the three entries directly above it (to the left, above, and to the right). The last remaining entry at the bottom of this pyramid is $a(n)$. The statement of the following theorem was conjectured in A359087 by Schott. It relates A359087 with A132894, which is defined as follows: For $n \in \mathbb{N}_0$, A132894(n) is the number of $(1,0)$ steps in all paths of length n with steps $U = (1,1), D = (1,-1)$, and $H = (1,0)$, starting at $(0,0)$, staying weakly above the x-axis.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$\text{A359087}(n) = n3^{n-1} - 2\text{A132894}(n-1). \tag{1}$$

Proof. The pyramid has n rows. Index the bottom row by 0 and index the rows above it increasingly. In particular, the upper row has index $n-1$. Index the elements in the upper row by

$$-n+1, -n+2, \ldots, -1, 0, 1, \ldots, n-2, n-1.$$

For an integer $-n+1 \leq j \leq n-1$, write $v_0(j)$ for the jth element in the upper row, i.e., $v_0(j) = n - |j|$. The key observation is that the entry $v_0(j)$ contributes its value to the bottom element exactly once for each path from $(0,0)$ to $(n-1, j)$, where the path uses only the steps $(1,1)$, $(1,0)$, and $(1,-1)$. For $m \in \mathbb{N}$ and $k \in \mathbb{Z}$, we denote by $T(m,k)$ the number of such paths of length m ending at (m,k). Thus, with the symmetry

$T(n-1,-j) = T(n-1,j)$, we obtain

$$\begin{aligned}
a(n) &= \sum_{j=-n+1}^{n-1} v_0(j)T(n-1,j) \\
&= nT(n-1,0) + \sum_{j=1}^{n-1}(n-j)(T(n-1,j) + T(n-1,-j)) \\
&= nT(n-1,0) + 2\sum_{j=1}^{n-1}(n-j)T(n-1,j) \\
&= n\left(T(n-1,0) + 2\sum_{j=1}^{n-1} T(n-1,j)\right) - 2\sum_{j=1}^{n-1} jT(n-1,j) \\
&= n\sum_{j=-n+1}^{n-1} T(n-1,j) - 2\sum_{j=1}^{n-1} jT(n-1,j).
\end{aligned}$$

Now, the expression

$$\sum_{j=-n+1}^{n-1} T(n-1,j)$$

merely counts all possible walks of length $n-1$, and their number is 3^{n-1}. For $m \in \mathbb{N}_0$, set

$$S_m = \sum_{j=1}^{m} jT(m,j).$$

It remains to show that $S_m =$ A132894(m). To this end, we first derive a recurrence for S_m. First, notice that $S_0 = 0$. Now, let $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. Using the recursion for $T(m,k)$,

$$T(m,k) = T(m-1,k-1) + T(m-1,k) + T(m-1,k+1),$$

we have

$$\begin{aligned}
S_m &= \sum_{j=1}^{m} jT(m,j)\\
&= \sum_{j=1}^{m} j(T(m-1,j-1)+T(m-1,j)+T(m-1,j+1))\\
&= \sum_{j=0}^{m-1}(j+1)T(m-1,j) + \sum_{j=1}^{m} jT(m-1,j) + \sum_{j=2}^{m+1}(j-1)T(m-1,j)\\
&= \left(S_{m-1} + \sum_{j=0}^{m-1} T(m-1,j)\right) + S_{m-1} + \left(S_{m-1} - \sum_{j=0}^{m-1} T(m-1,j) + T(m-1,0)\right)\\
&= 3S_{m-1} + T(m-1,0). \qquad (2)
\end{aligned}$$

Let $S(x) = \sum_{n=0}^{\infty} S_n x^n$ be the generating function of the S_n's and let $R(x) = \sum_{n=0}^{\infty} T(n,0)x^n$ be the generating function of the $T(n,0)$'s. Multiplying (2) by x^m and summing over $m \in \mathbb{N}$ yields

$$S(x) = 3xS(x) + xR(x).$$

Solving for $S(x)$ and using that

$$R(x) = \frac{1}{\sqrt{1-2x-3x^2}},$$

(e.g., A002426), we conclude that

$$S(x) = \frac{x}{(1-3x)\sqrt{1-2x-3x^2}},$$

which is precisely the generating function of the sequence A132894. In particular, $S_m = \text{A132894}(m)$, for every $m \in \mathbb{N}_0$. □

Corollary 1. *Let $A(x)$ be the generating function of the sequence* A359087*, i.e., $A(x) = \sum_{n=1}^{\infty} \text{A359087}(n)x^n$. Then*

$$A(x) = \frac{x}{(1-3x)^2} - \frac{2x^2}{(1+x)^{1/2}(1-3x)^{3/2}}.$$

Proof. Let $B(x)$ be the generating function of A132894, i.e., $B(x) = \sum_{n=1}^{\infty} \text{A132894}(n)x^n$. By (1),

$$A(x) = \frac{x}{(1-3x)^2} - 2xB(x).$$

It is well-known that

$$B(x) = \frac{x}{(1-3x)\sqrt{1-2x-3x^2}} = \frac{x}{(1+x)^{1/2}(1-3x)^{3/2}}.$$

It follows that

$$A(x) = \frac{x}{(1-3x)^2} - \frac{2x^2}{(1+x)^{1/2}(1-3x)^{3/2}},$$

as asserted. □

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.