Milan, December 2022

This document discusses the counting of distinct tilings of 5*n rectangles using squares of integer length sides. In particular, it focuses on “free” tilings, as opposed to “fixed” tilings.

The file https://oeis.org/A359019/a359019.pdf discusses rectangle tiling of smaller widths, and introduces the methods used here.

Recall in particular Formula 2 from that document:

free(n) = (fixed(n) + at_least_v(n) + at_least_h(n) + at_least_rot(n)) / 4

Counting free tilings of rectangles of width 5.

In this case, fixed(n) is given by sequence A054857, which has various formulae, one of which reads:

a(n) 2*a(n-1) +7*a(n-2) +6*a(n-3) -a(n-4) -5*a(n-5) -2*a(n-6) -3*a(n-7) -a(n-8)

for n >= 8.

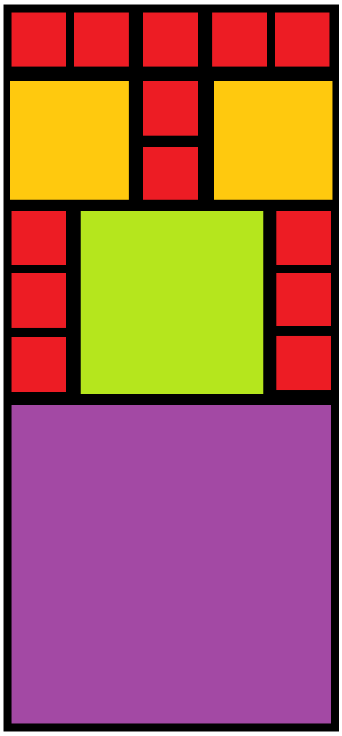
We will now try to find formulae for tilings possessing the 3 symmetries.

Tilings with vertical reflective symmetry.

It is clear that such tilings must consist of combinations of the following formations:

a. rows of 5 unit squares
b. a 2*2 square on the left, a vertical domino of 2 unit squares in the middle, and a 2*2 square on the right
c. a vertical strip of 3 unit squares on the left, a 3*3 square in the middle, and another vertical strip on the right
d. a 5*5 square

as shown in the following diagram:



For height n, therefore, the number of tilings with this symmetry is equal to the number of ways that n may be expressed as a sum of 1's, 2's, 3's and 5's, which is enumerated by sequence A079975, which has the following recursive formula:

$$a(n) = a(n-1)+a(n-2)+a(n-3)+a(n-5)$$

In other words:

$$\text{at\_least\_v}(n) = \text{A079975}(n)$$

Tilings with horizontal reflective symmetry.

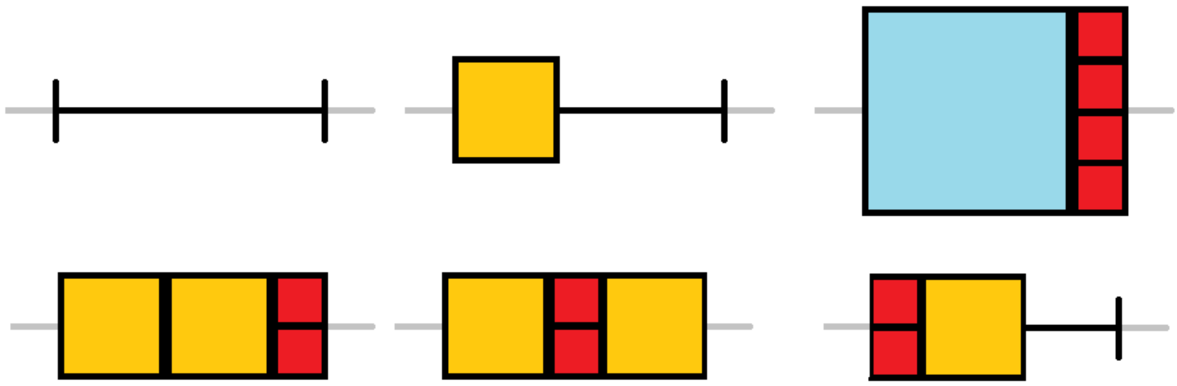
It is necessary to distinguish between rectangles of even and odd height.

1. Even height

It is clear that the axis of reflection must pass through one of the following combinations of tiles:

a. Through a clean separation of tiles above and below the axis;
b. Midway through a 2*2 square on the left, and on the right a clean separation of tiles above and below the axis;
c. As (b), the other way round;
d. Midway through a 4*4 square on the left and a vertical strip of 4 unit squares on the right;
e. As (d), the other way round;
f. Midway through a 2*2 square on the left; to its right another 2*2 square, and finally a vertical domino of 2 unit squares;
g. As (f) the other way round;
h. Midway through a 2*2 square on the left; to its right a vertical domino of 2 unit squares, and finally a 2*2 square;
i. Midway through a vertical domino of 2 unit squares; to its right a 2*2 square, and finally a clean separation of tiles above and below the axis;
j. As (i) the other way round.

Possibilities a, b, d, f, h and i are shown here:



Therefore, for even n:

at_least_h(n) = A054857(n/2)	(from a)
+ 2* fixed_md(n/2)	(from b,c)
+ 2*A054857((n-4)/2)	(from d,e)
+ 3*A054857((n-2)/2)	(from f,g,h)
+ 2* fixed_mt(n/2)	(from i,j)

This formula makes reference to fixed_md and fixed_mt which need to be defined.

fixed_md(n) gives the number of fixed tilings of a width 5 rectangle of height n with a horizontal domino missing from one of its corners. “Missing domino” – hence “md”.

fixed_mt(n) gives the number of fixed tilings of a width 5 rectangle of height n with a horizontal tromino missing from one of its corners. “Missing tromino” – hence “mt”.

Calculation of fixed_md(n)

The tiling could be performed in various ways. In the diagrams, the 2 black squares represent the missing domino.

i. 3 unit squares on the same row as the missing domino

This case will contribute fixed(n-1) to fixed_md(n), where fixed(n) = A054857(n).

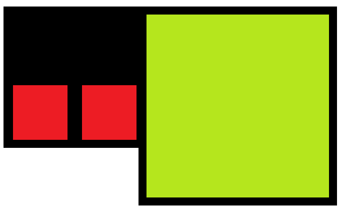
ii. A 2*2 square and a vertical domino of 2 unit squares

This case will contribute fixed_mt(n-1) to fixed_md(n).

iii. A unit square and then a 2*2 square

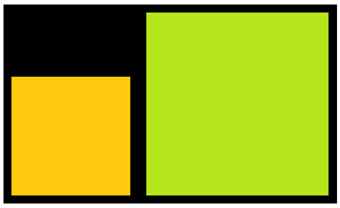
This case will contribute fixed_md(n-1) to fixed_md(n).

iv. A 3*3 square on the right, and a domino of 2 unit squares below the missing domino



This case will contribute fixed_mt(n-2) to fixed_md(n).

v. A 3*3 square on the right, and a 2*2 square below the missing domino



This case will contribute A054857(n-3) to fixed_md(n).

In summary:

fixed_md(1)=1
fixed_md(2)=3
for n > 2:
fixed_md(n) = A054857(n-1) + fixed_mt(n-1) + fixed_md(n-1) + fixed_mt(n-2) + A054857(n-3)

Calculation of fixed_mt(n)

The tiling could be performed in various ways. In the diagrams, the 3 black squares represent the missing tromino.

i. 2 unit squares on the same row as the missing tromino

This case will contribute fixed(n-1) to fixed_mt(n), where fixed(n) = A054857(n).

ii. A 2*2 square to the right of the missing tromino

This case will contribute fixed_md(n-1) to fixed_mt(n).

In summary:

fixed_mt(1)=1
For n>1,

fixed_mt(n)= A054857(n-1) + fixed_md(n-1)

We can therefore simplify fixed_md as follows:

fixed_md(1)=1
fixed_md(2)=3
fixed_md(3)=15
for n > 3:

$$fixed\_md(n) = A054857(n-1) + A054857(n-2) + fixed\_md(n-2) + fixed\_md(n-1) + 2*A054857(n-3) + fixed\_md(n-3)$$

And

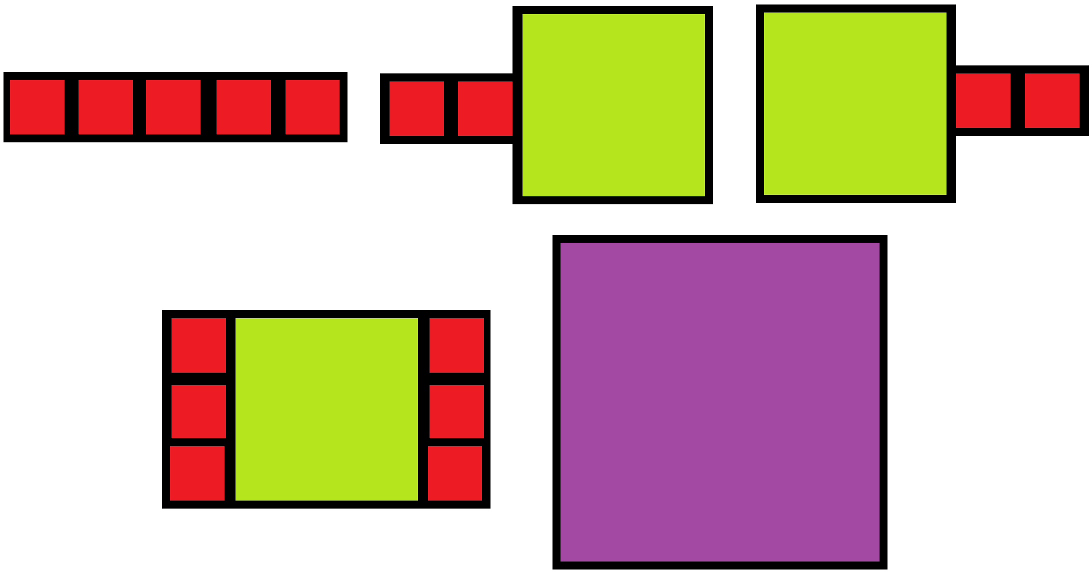
$$at\_least\_h(n) = A054857(n/2) + 2*fixed\_md(n/2) + 2*A054857((n-4)/2) + 3*A054857((n-2)/2) + 2*(A054857((n/2)-1) + fixed\_md((n/2)-1))$$

2. Odd height

The axis of reflection must pass through one of the following combinations of tiles:

a. A row of 5 unit squares
b. A 3*3 square followed by a horizonal domino of 2 unit squares
c. As (b) but the other way round
d. A 3*3 square in the middle, flanked on both sides by a vertical strip of 3 unit squares
e. A 5*5 square

These possibilities are shown here:



Therefore, for odd n, following the same logic as used above:

$$at\_least\_h(n) = A054857((n-1)/2) + 2*fixed\_mt((n-1)/2) + A054857((n-3)/2) + A054857((n-5)/2)$$

and so:

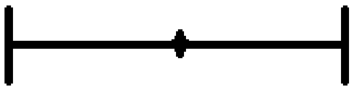
$$at\_least\_h(n) = A054857((n-1)/2) + 3*A054857((n-3)/2) + 2*fixed\_md((n-3)/2) + A054857((n-5)/2)$$

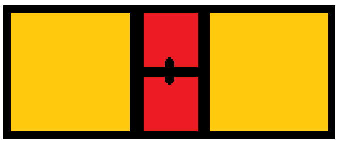
Tilings with 180 degree rotational symmetry.

1. Even height

The centre of rotation must be in one of the following positions:

i. on a line that separates cleanly the tiles above and below it;
ii. at the centre of a vertical domino flanked by two 2*2 squares.





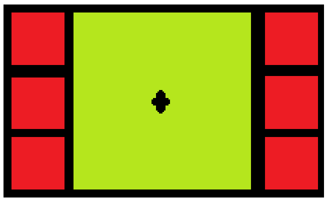
Therefore, for even n:

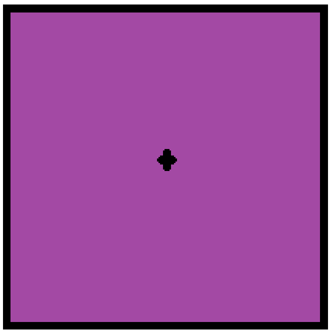
at_least_rot(n) = A054857(n/2) + A054857((n-2)/2)

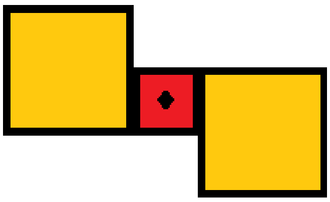
2. Odd height

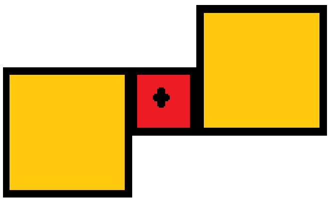
The centre of rotation must be in one of the following formations:

a) The centre of a row of 5 unit squares
b) The centre of a 3*3 square flanked by vertical strips of 3 unit squares
c) The centre of a 5*5 square
d) The centre of a unit square flanked on the left by a 2*2 square slightly above, and on the right by a 2*2 square slightly below
e) As (d) but the other way round









Therefore, for odd n:

at_least_rot(n) = A054857 ((n-1)/2) + A054857 ((n – 3)/2) + A054857 ((n – 5)/2) + 2*fixed_md((n-1)/2)

Conclusion.

For even n, recalling Formula 2:

free(n) = (A054857(n) + A079975(n) + A054857(n/2) + 2* fixed_md(n/2) + 2*A054857((n-4)/2) + 3*A054857((n-2)/2) + 2* (A054857((n/2)-1) + fixed_md((n/2)-1)) + A054857(n/2) + A054857((n-2)/2))/4

where fixed_md as above.

And therefore, for even n:

free(n) = (A054857(n) + A079975(n) + 2*A054857(n/2) + 2* fixed_md(n/2) + 2*A054857((n-4)/2) + 4*A054857((n-2)/2) + 2* (A054857((n/2)-1) + fixed_md((n/2)-1)))/4

For odd n:

free(n) = (A054857(n) + A079975(n) + A054857((n-1)/2) + 3*A054857((n-3)/2) + 2*fixed_md((n-3)/2) + A054857((n-5)/2) + A054857 ((n-1)/2) + A054857 ((n – 3)/2) + A054857 ((n – 5)/2) + 2*fixed_md((n-1)/2))/4

And therefore, for odd n:

free(n) = (A054857(n) + A079975(n) + 2*A054857((n-1)/2) + 4*A054857((n-3)/2) + 2*fixed_md((n-3)/2) + 2*A054857((n-5)/2) + 2*fixed_md((n-1)/2))/4

Footnote:

- The formulae are valid only for n > 5

John Mason

TheIllustratedPolyomino (at) gmail (dot) com