Notes on A357563

Peter Bala, Oct 16 2022

Let A(n) = A357563(n). The definition is

$$A(n) = a(n) - 2a(a(a(n))), \text{ for } n \geq 3, \tag{1}$$

where $a(n) = \text{A356988}(n)$ is defined by the nested recurrence

$$a(n) = n - a(a(n - a(a(a(n-1))))) \tag{2}$$

with the initial condition a(1) = 1.

Definition. The sequence $\{u(n)\}$ is **slow** if $u(n+1) - u(n) \in \{0, 1\}$.

It is shown in [1] that both the sequences $\{a(n)\}$ and $\{a(a(a(n)))\}$ are slow. It follows from the definition of $A(n)$ in (1) that the difference $A(n+1) - A(n) \in \{-1, 0, 1\}$ (we don't get the value $A(n+1) - A(n) = -2$ since this would require $a(n+1) - a(n) = 0$ and $a(a(a(n+1))) - a(a(a(n))) = 1$, which is clearly not possible).

In order to analyse the structure of the sequence $\{A(n)\}$ we will need following facts about A356988 [1, Proposition 1]:

a) for $k >= 0$,

$$a(\mathrm{L}(k+1) + j) = \mathrm{F}(k+2) \text{ for } 0 \leq j \leq \mathrm{F}(k-1), \tag{3}$$

where $\mathrm{F}(n) = \text{A000045}(n)$, the n-th Fibonacci number and $\mathrm{L}(n) = \text{A000032}(n)$, the n-th Lucas number (recall that $\mathrm{L}(n) = \mathrm{F}(n+1) + \mathrm{F}(n-1)$).

b) for $k >= 2$,

$$a(\mathrm{F}(k+1)) = \mathrm{F}(k). \tag{4}$$

In addition, we will require the pair of results

$$\mathrm{a}(2\mathrm{F}(k)) = \mathrm{L}(k-1) \text{ for } k \geq 2 \tag{5}$$

$$\mathrm{a}(3\mathrm{F}(k)) = 2\mathrm{F}(k) \text{ for } k \geq 1, \tag{6}$$

which follow easily from [1, Proposition 2].

The structure of $\{A(n)\}$.

1) Ascent to plateau: Assume $k \geq 2$. On the interval $[3\mathrm{F}(k), \mathrm{L}(k+1)]$ the line graph of the sequence ascends with slope 1 from a value of 0 at abscissa $n = 3\mathrm{F}(k)$ to a value $\mathrm{F}(k-1)$ at abscissa value $n = \mathrm{L}(k+1)$.

Proof. At the left endpoint of the interval we calculate

$$\begin{aligned} A\left(3\mathrm{F}(k)\right) &= a\left(3\mathrm{F}(k)\right) - 2a\left(a\left(a\left(3\mathrm{F}(k)\right)\right)\right) \\ &= 2\mathrm{F}(k) - 2a\left(a\left(2\mathrm{F}(k)\right)\right) \text{ by } (6) \\ &= 2\mathrm{F}(k) - 2a\left(\mathrm{L}(k-1)\right) \text{ by } (5) \\ &= 2F(k) - 2\mathrm{F}(k) \text{ by } (3) \\ &= 0. \end{aligned} \tag{7}$$

At the right endpoint of the interval we calculate

$$\begin{aligned} A\left(\mathrm{L}(k+1)\right) &= a\left(\mathrm{L}(k+1)\right) - 2a\left(a\left(a\left(\mathrm{L}(k+1)\right)\right)\right) \\ &= \mathrm{F}(k+2) - 2a\left(a\left(\mathrm{F}(k+2)\right)\right) \text{ by } (3) \\ &= \mathrm{F}(k+2) - 2a\left(\mathrm{F}(k+1)\right) \text{ by } (4) \\ &= \mathrm{F}(k+2) - 2\mathrm{F}(k) \text{ by } (4) \\ &= \mathrm{F}(k-1). \end{aligned}$$

Therefore, on the integer interval $[3\mathrm{F}(k), \mathrm{L}(k+1)]$, of length $\mathrm{L}(k+1) - 3\mathrm{F}(k) = \mathrm{F}(k-1)$, the sequence increases in value from 0 to $F(k-1)$. Since, as shown above, $A(n+1) - A(n) \in \{-1, 0, 1\}$, it follows that on the interval $[3\mathrm{F}(k), L(k+1)]$ the line graph of the sequence has slope 1. $\square$

2) Plateau: on the integer interval $[\mathrm{L}(k+1), \mathrm{F}(k+3)]$ of length $\mathrm{F}(k-1)$ the sequence has the constant value $\mathrm{F}(k-1)$.

Proof. We calculate for $0 \leq j \leq \mathrm{F}(k-1)$

$$\begin{aligned} A\left(\mathrm{L}(k+1) + j\right) &= a\left(\mathrm{L}(k+1) + j\right) - 2a\left(a\left(a\left(\mathrm{L}(k+1) + j\right)\right)\right) \\ &= \mathrm{F}(k+2) - 2a\left(a\left(\mathrm{F}(k+2)\right)\right) \text{ by } (3) \\ &= \mathrm{F}(k+2) - 2\mathrm{F}(k) \text{ applying (4) twice} \\ &= \mathrm{F}(k-1). \square \end{aligned}$$

3) Descent to 0: finally, on the integer interval $[\mathrm{F}(k+3), 3\mathrm{F}(k+1)]$, of length $3\mathrm{F}(k+1) - \mathrm{F}(k+3) = \mathrm{F}(k-1)$, the sequence decreases in value from $F(k-1)$ to $A(3\mathrm{F}(k+1)) = 0$, by (7). Again, since $A(n+1) - A(n) \in \{-1, 0, 1\}$, it follows that on the interval $[\mathrm{F}(k+3), 3\mathrm{F}(k+1)]$ the line graph of the sequence has slope -1.

References

[1] Peter Bala, Notes on A356988