Notes on A357562

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Let A(n) = A357562(n). The definition is

$$A(n) = n - 2a(a(n)) \text{ for } n \geq 2, \tag{1}$$

where $a(n) = \text{A356988}(n)$ is defined by the nested recurrence

$$a(n) = n - a(a(n - a(a(a(n-1))))) \tag{2}$$

with the initial condition a(1) = 1.

Definition. The sequence $\{u(n)\}$ is **slow** if $u(n+1) - u(n) \in \{0, 1\}$. It is shown in [1] that the sequences $\{a(n)\}$ and $\{a(a(n))\}$ are slow.

From (1) we have $A(n+1) - A(n) = 1 - 2(\ a(a(n+1)) - a(a(n))\ )$. Since the sequence $\{a(a(n)\}$ is slow it follows that the difference $A(n+1) - A(n)$ is either 1 or -1.

In order to analyse the structure of the sequence $\{A(n)\}$ we will need following facts about A356988 [1, Proposition 2]:

a) for $k >= 2$,

$$a(\text{L}(k-1)) = \text{F}(k), \tag{3}$$

where $\text{F}(n) = \text{A000045}(n)$, the n-th Fibonacci number and $\text{L}(n) = \text{A000032}(n)$, the n-th Lucas number (recall that $\text{L}(n) = \text{F}(n+1) + \text{F}(n-1)$).

b) for $k >= 1$,

$$a(\text{F}(k+1)) = \text{F}(k). \tag{4}$$

In addition, we will require the result

$$\text{a}(2\text{F}(k)) = \text{L}(k-1) \text{ for } k \geq 2, \tag{5}$$

which follows easily from Proposition 2 of [1] on writing $2\text{F}(k)$ as $\text{F}(k+1) + \text{F}(k-2)$.

The structure of $\{A(n)\}$.

The sequence vanishes at abscissa values $n = 2, 4, 6, 10, 16, 26, ..., 2\text{F}(k), ...$ The graph of the sequence, starting from the zero value at abscissa $n = 2\text{F}(k)$, ascends with slope 1 to a local peak at height $\text{F}(k-1)$ at abscissa value $n = \text{F}(k+2)$ before descending with slope -1 to the next zero at at abscissa $n = 2\text{F}(k+1)$.

Proof. Assume $k \geq 2$.

1) First we prove the claim that oOn the interval $[2\mathrm{F}(k), \mathrm{F}(k+2)]$ the line graph of the sequence ascends with slope 1 from a value of 0 at abscissa $n = 2\mathrm{F}(k)$ to a local peak at height $\mathrm{F}(k-1)$ at abscissa value $n = \mathrm{F}(k+2)$.

At the left endpoint of the interval we calculate

$$\begin{aligned} A\,(2\mathrm{F}(k)) &= 2\mathrm{F}(k) - 2a\,(a\,(2\mathrm{F}(k))) \\ &= 2\mathrm{F}(k) - 2a\,(\mathrm{L}(k-1)) \text{ by (5)} \\ &= 2\mathrm{F}(k) - 2\mathrm{F}(k) \text{ by (3)} \\ &= 0. \end{aligned} \tag{6}$$

At the right endpoint of the interval we calculate

$$\begin{aligned} A\,(\mathrm{F}(k+2)) &= \mathrm{F}(k+2) - 2a\,(a\,(\mathrm{F}(k+2))) \\ &= \mathrm{F}(k+2) - 2a\,(\mathrm{F}(k+1)) \text{ by (4)} \\ &= \mathrm{F}(k+2) - 2\mathrm{F}(k) \text{ again by (4)} \\ &= \mathrm{F}(k-1). \end{aligned}$$

Thus on the integer interval $[2\mathrm{F}(k), \mathrm{F}(k+2)]$, of length $\mathrm{F}(k+2) - 2\mathrm{F}(k) = \mathrm{F}(k-1)$, the sequence $\{A(n)\}$ increases in value from 0 to $F(k-1)$. But we showed above that $A(n+1) - A(n)$ is either 1 or -1. Hence on the interval $[2\mathrm{F}(k), \mathrm{F}(k+2)]$ the line graph of the sequence must have slope 1.

2) On the interval $[\mathrm{F}(k+2), 2\mathrm{F}(k+1)]$, of length $2\mathrm{F}(k+1) - \mathrm{F}(k+2) = \mathrm{F}(k-1)$, the sequence $\{A(n)\}$ decreases in value from a peak value of $\mathrm{F}(k-1)$ down to the value $A(2\mathrm{F}(k+1)) = 0$ by (6). Again, since $A(n+1) - A(n)$ is either 1 or -1, it must be the case that on the interval $[\mathrm{F}(k+2), 2\mathrm{F}(k+1)]$ the line graph of the sequence has slope -1.

References

[1] Peter Bala, Notes on A356988