Notes on A356988

Peter Bala, Sep 27 2022

$a(n) = \text{A356988}(n)$ is defined by the nested recurrence

$$a(n) = n - a(a(n - a(a(a(n-1))))) \qquad (1)$$

with the initial condition a(1) = 1.

If we define the square $a^{(2)}$ and the cube $a^{(3)}$ of the sequence $a = \{a(n) : n \geq 1\}$ by $a^{(2)} = \{a(a(n)) : n \geq 1\}$ and $a^{(3)} = \{a(a(a(n))) : n \geq 1\}$ then the recurrence (1) can be rewritten as

$$a(n) = n - a^{(2)}(n - a^{(3)}(n-1)). \qquad (2)$$

An easy induction argument shows that

$$1 \leq a(n) < n \text{ for } n \geq 2. \qquad (3)$$

A table of the first few terms of A356988 is shown below. The graph of the sequence appears to consist of a series of plateaus joined by lines of slope 1. Our aim in these notes is to prove the following more precise results completely determining the structure of the sequence:

(i) the plateaus start at abscissa values (shown in red in the table) $n = 4, 7, 11, 18, 29, \ldots$, given by the sequence of Lucas numbers $(\mathrm{L}(k))_{k>=3}$, and end at abscissa values (shown in green in the table) $n = 5, 8, 13, 21, 34, \ldots$, given by the Fibonacci sequence $(\mathrm{F}(k+2))_{k\geq 3}$.

(ii) the plateaus are at heights (shown in blue in the table) $3, 5, 8, 13, 21, \ldots$, given by the Fibonacci sequence $(\mathrm{F}(k+1))_{k\geq 3}$.

(iii) the plateaus are joined by lines of slope 1.

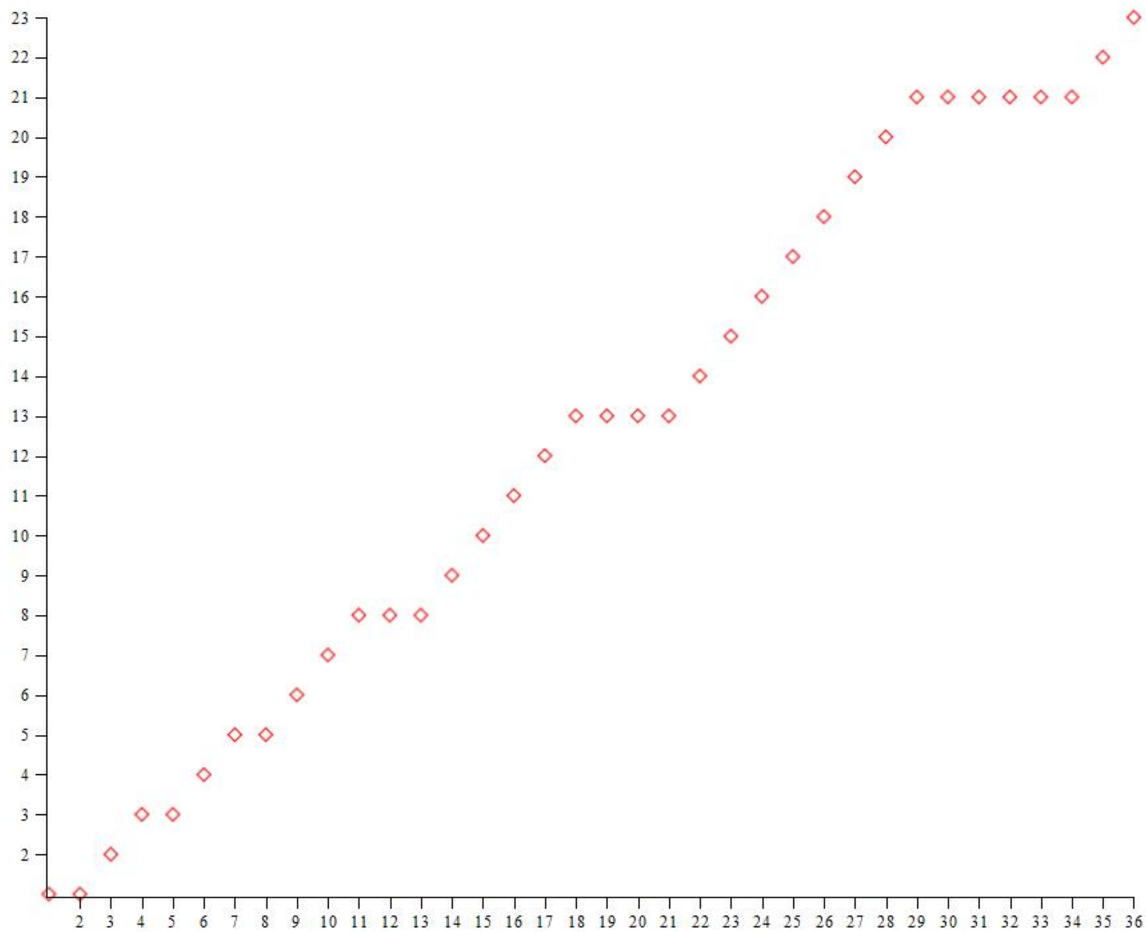
Table: $a(n)$ for $n = 1..36$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$a(n)$	1	1	2	3	3	4	5	5	6	7	8	8	8	9

n	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$a(n)$	10	11	12	13	13	13	13	14	15	16	17	18	19	20

n	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$a(n)$	21	21	21	21	21	21	22	23	24	25	26	27	28	29

Graph of $a(n) : n = 1..36$



First we show that the sequence is monotone and slow-growing.

Definition. A sequence $\{u(n) : n \geq 1\}$ is *slow* if $u(n+1) - u(n) = \epsilon$ for $n \geq 1$, where the symbol ϵ denotes a quantity that only takes on the value 0 or 1. Note that the expression $1 - \epsilon$ also has the value either 0 or 1.

Proposition 1. A356988 *is slow: for* $n \geq 1$

$$a(n+1) - a(n) = \epsilon\,. \tag{4}$$

Proof. By strong induction. By inspection, (4) holds for $n = 1$. Assume (4) holds up to $n = N$, that is,

$$a(k+1) - a(k) = \epsilon,\ 1 \le k \le N. \tag{5}$$

It follows from (3) and the inductive hypothesis (5) that

$$a^{(2)}(k+1) - a^{(2)}(k) = \epsilon,\ 1 \le k \le N \tag{6}$$

and hence also

$$a^{(3)}(k+1) - a^{(3)}(k) = \epsilon,\ 1 \le k \le N. \tag{7}$$

In particular from (7)

$$a^{(3)}(N+1) - a^{(3)}(N) = \epsilon \tag{8}$$

and therefore

$$\begin{aligned} \left(N+2-a^{(3)}(N+1)\right) - \left(N+1-a^{(3)}(N)\right) &= 1 - \left(a^{(3)}(N+1) - a^{(3)}(N)\right) \\ &= 1-\epsilon \text{ by (8)} \\ &= \epsilon. \end{aligned} \tag{9}$$

Set $k = N+1-a^{(3)}(N)$. We see from (3) that $1 \le k \le N$. It then follows from (6) and (9) that

$$a^{(2)}\left(N+2-a^{(3)}(N+1)\right) - a^{(2)}\left(N+1-a^{(3)}(N)\right) = \epsilon. \tag{10}$$

Therefore, from the defining recurrence (2),

$$\begin{aligned} a(N+2) - a(N+1) &= 1 - a^{(2)}\left(N+2-a^{(3)}(N+1)\right) + a^{(2)}\left(N+1-a^{(3)}(N)\right) \\ &= 1-\epsilon \end{aligned}$$

by (10). Since $1-\epsilon = \epsilon$, this completes the inductive argument. □

The structure of A356988 Let $\mathrm{F}(n) = \mathrm{A000045}(n)$ denote the n-th Fibonacci number (we will also require the value $\mathrm{F}(-1) = 1$). Let $\mathrm{L}(n) = \mathrm{A000032}(n)$ denote the n-th Lucas number. Recall that $\mathrm{L}(n) = \mathrm{F}(n+1) + \mathrm{F}(n-1)$.

Proposition 2. The following hold for $n >= 2$:

(i) on the integer interval $[\mathrm{L}(n-1), \mathrm{F}(n+1)]$ of length $\mathrm{F}(n-3)$ the sequence has the constant value $\mathrm{F}(n)$

(ii) on the integer interval $[\mathrm{F}(n+1), \mathrm{L}(n)]$ of length $\mathrm{F}(n-1)$ the graph of the sequence has slope 1.

Proof. The proof is by a layered induction argument requiring secondary induction arguments within the main induction argument. Let $k \geq 2$. We define the pair of statements $\mathrm{P}(k)$ and $\mathrm{R}(k)$ as follows:

$$a(\mathrm{L}(k-1)+j) = \mathrm{F}(k) \text{ holds for } 0 \leq j \leq \mathrm{F}(k-3) \quad [\mathrm{P}(k)]$$

$$a(\mathrm{F}(k+1)+j) = \mathrm{F}(k)+j \text{ holds for } 0 \leq j \leq \mathrm{F}(k-1) \quad [\mathrm{R}(k)]$$

and make the inductive hypothesis that the statements $\mathrm{P}(n-1)$, $\mathrm{R}(n-1)$, $\mathrm{R}(n)$ and $\mathrm{P}(n)$ are true for some $n \geq 3$. The base case when $n=3$ is easily checked. We wish to prove that $\mathrm{P}(n+1)$ and $\mathrm{R}(n+1)$ are true. First we prove $\mathrm{P}(n+1)$, that is,

$$a(\mathrm{L}(n)+j) = \mathrm{F}(n+1) \text{ holds for } 0 \leq j \leq \mathrm{F}(n-2) \quad [\mathrm{P}(n+1)]$$

by a secondary induction on j.

The base case $j=0$ of $\mathrm{P}(n+1)$ says that $a(\mathrm{L}(n)) = \mathrm{F}(n+1)$, but this is simply the final case $j = \mathrm{F}(n-1)$ in $\mathrm{R}(n)$. Suppose now $a(\mathrm{L}(n)+j) = \mathrm{F}(n+1)$ holds for some j in the range $0 \leq j < \mathrm{F}(n-2)$.

Then by (2),

$$\begin{aligned}
a(\mathrm{L}(n)+j+1) &= \mathrm{L}(n)+j+1-a^{(2)}\left(L(n)+j+1-a^{(3)}(\mathrm{L}(n)+j)\right) \\
&= \mathrm{L}(n)+j+1-a^{(2)}\left(j+1+\mathrm{L}(n)-a^{(2)}(\mathrm{F}(n+1))\right) \\
&= \mathrm{L}(n)+j+1-a^{(2)}(j+1+\mathrm{L}(n)-\mathrm{F}(n-1)) \text{ by } \mathrm{R}(n) \text{ and } \mathrm{R}(n-1) \\
&= \mathrm{L}(n)+j+1-a^{(2)}(j+1+\mathrm{F}(n+1)) \\
&= \mathrm{L}(n)+j+1-(j+1+\mathrm{F}(n-1)) \text{ by } \mathrm{R}(n) \text{ and } \mathrm{R}(n-1) \\
&= \mathrm{F}(n+1)
\end{aligned}$$

and the induction goes through. This completes the proof of $\mathrm{P}(n+1)$.

We now turn our attention to proving $\mathrm{R}(n+1)$, which states that on the interval $[\mathrm{F}(n+2), \mathrm{L}(n+1)]$ of length $\mathrm{F}(n)$ the line graph of the sequence has slope 1, that is,

$$a(\mathrm{F}(n+2)+j) = \mathrm{F}(n+1)+j \text{ holds for } 0 \leq j \leq \mathrm{F}(n) \quad [\mathrm{R(n+1)}].$$

The proof is in three stages, each stage requiring an induction argument on j. Since $\mathrm{F}(n) = \mathrm{F}(n-2)+\mathrm{F}(n-3)+\mathrm{F}(n-2)$, the interval $[\mathrm{F}(n+2), \mathrm{L}(n+1)]$ can be split into three subintervals $[\mathrm{F}(n+2), \mathrm{F}(n+2)+\mathrm{F}(n-2)]$, $[\mathrm{F}(n+2)+\mathrm{F}(n-2), \mathrm{F}(n+2)+\mathrm{F}(n-1)]$ and $[\mathrm{F}(n+2)+\mathrm{F}(n-1), \mathrm{L}(n+1)]$.

Stage 1: On the first subinterval $[\mathrm{F}(n+2), \mathrm{F}(n+2)+\mathrm{F}(n-2)]$ of length $\mathrm{F}(n-2)$ we make the inductive hypothesis that

$$a(\mathrm{F}(n+2)+j) = \mathrm{F}(n+1)+j \quad \text{(H1)}$$

holds for some j in the range $0 \le j < \mathrm{F}(n-2)$. The base case $j = 0$ for the induction is simply the final case $j = \mathrm{F}(n-2)$ of $\mathrm{P}(n+1)$.

Then by (2),

$$\begin{aligned}
a(\mathrm{F}(n+2)+j+1) &= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a^{(3)}\left(\mathrm{F}(n+2)+j\right)\right) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+1)+j\right)\right) \text{ by H1} \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a(\mathrm{F}(n)+j)\right) \text{ by R}(n) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-(\mathrm{F}(n-1)+j)\right) \text{ by R}(n-1) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(1+\mathrm{F}(n+2)-\mathrm{F}(n-1)\right) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+1)+1+\mathrm{F}(n-2)\right) \\
&= \mathrm{F}(n+2)+j+1-a\left(\mathrm{F}(n)+1+\mathrm{F}(n-2)\right) \text{ by R}(n) \\
&= \mathrm{F}(n+2)+j+1-a\left(\mathrm{L}(n-1)+1\right) \\
&= \mathrm{F}(n+2)+j+1-\mathrm{F}(n) \text{ by P}(n) \\
&= \mathrm{F}(n+1)+j+1
\end{aligned}$$

thus completing the induction argument on the first subinterval.

Stage 2. On the second subinterval $[\mathrm{F}(n+2)+\mathrm{F}(n-2), \mathrm{F}(n+2)+\mathrm{F}(n-1)]$ of length $\mathrm{F}(n-3)$ we make the inductive hypothesis that

$$a(\mathrm{F}(n+2)+j) = \mathrm{F}(n+1)+j \quad \text{(H2)}$$

holds for some j in the range $\mathrm{F}(n-2) \le j < \mathrm{F}(n-1)$. The base case for the induction when $j = \mathrm{F}(n-2)$ is simply the final case of H1 established above in stage 1. Define j' by $j = \mathrm{F}(n-2)+j'$, so that $0 \le j' < \mathrm{F}(n-1)-\mathrm{F}(n-2)$, that is, $0 \le j' < \mathrm{F}(n-3)$.

Then by (2),

$$\begin{aligned}
a(\mathrm{F}(n+2)+j+1) &= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a^{(3)}\left(\mathrm{F}(n+2)+j\right)\right) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+1)+j\right)\right) \text{ by H2} \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a(\mathrm{F}(n)+j)\right) \text{ by R}(n) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-a\left(\mathrm{L}(n-1)+j'\right)\right) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+2)+j+1-\mathrm{F}(n)\right) \text{ by P}(n) \\
&= \mathrm{F}(n+2)+j+1-a^{(2)}\left(\mathrm{F}(n+1)+j+1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{F}(n+2)+j+1 - a\left(\mathrm{F}(n)+j+1\right) \text{ by } \mathrm{R}(n)\\
&= \mathrm{F}(n+2)+j+1 - a\left(\mathrm{F}(n)+\mathrm{F}(n-2)+j'+1\right)\\
&= \mathrm{F}(n+2)+j+1 - a\left(\mathrm{L}(n-1)+j'+1\right)\\
&= \mathrm{F}(n+2)+j+1 - \mathrm{F}(n) \text{ by } \mathrm{P}(n)\\
&= \mathrm{F}(n+1)+j+1
\end{aligned}
$$

thus completing the induction argument on the second subinterval.

Stage 3. On the third subinterval $[\mathrm{F}(n+2)+F(n-1), \mathrm{L}(n+1)]$ of length $\mathrm{F}(n-2)$ the inductive hypothesis is that

$$a(\mathrm{F}(n+2)+j) = \mathrm{F}(n+1)+j \quad \text{(H3)}$$

holds for some j in the range $\mathrm{F}(n-1) \le j < \mathrm{L}(n+1) - \mathrm{F}(n+2)$, that is, for some j in the range $\mathrm{F}(n-1) \le j < \mathrm{F}(n)$.

Define j' by $j = \mathrm{F}(n-1)+j'$, so that $0 \le j' < \mathrm{F}(n) - \mathrm{F}(n-1)$, that is, $0 \le j' < \mathrm{F}(n-2)$.

The base case for the induction when $j' = 0$ is simply the final case of H2 established in stage 2 above.

Then by (2),

$$
\begin{aligned}
a(\mathrm{F}(n+2)+j+1) &= \mathrm{F}(n+2)+j+1 - a^{(2)}\Big(\mathrm{F}(n+2)+j+1 - a^{(3)}\left(\mathrm{F}(n+2)+j\right)\Big)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\Big(\mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{F}(n+1)+j\right)\Big) \text{ by H3}\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\Big(\mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{F}(n+1)+\mathrm{F}(n-1)+j'\right)\Big)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\Big(\mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{L}(n)+j'\right)\Big)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{F}(n+2)+j+1 - a(\mathrm{F}(n+1))\right) \text{by } \mathrm{P}(n+1)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{F}(n+2)+j+1 - \mathrm{F}(n)\right) \text{by } \mathrm{R}(n)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{F}(n+1)+j+1\right)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{F}(n+1)+\mathrm{F}(n-1)+j'\right)\\
&= \mathrm{F}(n+2)+j+1 - a^{(2)}\left(\mathrm{L}(n)+j'\right)\\
&= \mathrm{F}(n+2)+j+1 - a\left(\mathrm{F}(n+1)\right) \text{ by } \mathrm{P}(n+1)\\
&= \mathrm{F}(n+2)+j+1 - \mathrm{F}(n) \text{ by } \mathrm{R}(n)\\
&= \mathrm{F}(n+1)+j+1.
\end{aligned}
$$

This finishes the proof of $\mathrm{R}(n+1)$ and completes the proof of the main induction argument, thus establishing the proposition. □

Related slow sequences. The following statements about slow sequences are easily proved.

If the sequences $\{u(n) : n \geq 1\}$ and $\{v(n) : n \geq 1\}$ are both slow then

(i) the sequence $\{n - u(n)\}$ is slow.

(ii) consequently, the sequences $\{n - u(n - u(n))\}$, $\{n - u(n - u(n - u(n)))\}$, ..., are all slow.

(iii) the iterated sequences $\{u(u(n))\}$, $\{u(u(u(n)))\}$, ... are slow.

(iv) the sequence $u(n) + v(n - u(n))$ is slow.

(v) the composition sequence $\{u(v(n))\}$ is slow.

Starting with the slow sequence $\{a(n)\}$ we can use these observations to construct several other slow sequences of interest. We finish with a few examples.

Examples

1. The sequence $\{a(n) + a(n - a(n)) : n \geq 2\}$ submitted as A356991 is slow.

The first few terms of the sequence are 2, 3, 4, 4, 5, 6, 7, 8, 9, 10, 11, 11, 12, 13, 14, 15, 16, 17, 18, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 29, 29, 30, 31, ... The graph of the sequence has plateaus at heights 4, (7), 11, 18, 29, ..., the Lucas numbers.

2. The sequence $\{n - a(n - a(n - a(n - a(n - a(n - a(n)))))) : n \geq 2\}$ submitted as A356992 is slow.

The first few terms of the sequence are 1, 2, 3, 4, 4, 4, 5, 6, 7, 7, 7, 8, 9, 10, 11, 11, 11, 11, 12, 13, 14, 15, 16, 17, 18, 18, 18, 18, 18, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 29, 29, 29, 29, 29, 29, 29, 29, 30, ... The graph of the sequence has plateaus at heights $4, 7, 11, 18, 29, ...$, conjecturally the Lucas numbers.

3. The sequence $\{a(n - a(n - a(n - a(n - a(n))))) : n \geq 2\}$ submitted as A356993 is slow.

The first few terms of the sequence are 1, 1, 2, 3, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 8, 8, 8, 8, 8, 9, 10, 11, 11, 12, 13, 13, 13, 13, 13, 13, 13, 14, 15, 16, 17, 18, 18, 18, 19, 20, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 21, 22, 23, 24, 25, 26, 27, 28, 29, 29, 29, 29, 30, 31,.... The heights of the plateaus beginning 3, (4), 5, 7, 8, 11, 13, 21, 29, ... appear to be alternately Fibonacci numbers and Lucas numbers.

4. The sequence $\{n - a^3(n) : n \geq 1\}$ submitted as A356994 is slow.

The first few terms of the sequence are 0, 1, 2, 3, 4, 4, 5, 6, 6, 7, 8, 9, 10, 10, 10, 11, 12, 13, 14, 15, 16, 16, 16, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 26, 26, 26, 26, 26, 27, ... The graph of the sequence has plateaus at heights $4, 6, 10, 16, 26, ...$, double the Fibonacci numbers.