# Proof that A355641 consists of all numbers divisible by at least one of 5, 6, 8, 9, 14, 21.

If $k$ is divisible by 5, then $k$ is in the sequence because $k = \frac{k}{5} + \frac{k}{5} + \frac{k}{5} + \frac{k}{5} + \frac{k}{5}$ exhibits $k$ as the sum of 5 divisors of $k$ and thus a member of A355641.

If $k$ is divisible by 6, then $k = \frac{k}{3} + \frac{k}{6} + \frac{k}{6} + \frac{k}{6} + \frac{k}{6}$.

If $k$ is divisible by 8, then $k = \frac{k}{4} + \frac{k}{4} + \frac{k}{4} + \frac{k}{8} + \frac{k}{8}$.

If $k$ is divisible by 9, then $k = \frac{k}{3} + \frac{k}{3} + \frac{k}{9} + \frac{k}{9} + \frac{k}{9}$.

If $k$ is divisible by 14, then $k = \frac{k}{2} + \frac{k}{7} + \frac{k}{7} + \frac{k}{7} + \frac{k}{14}$.

If $k$ is divisible by 21, then $k = \frac{k}{3} + \frac{k}{3} + \frac{k}{7} + \frac{k}{7} + \frac{k}{21}$.

Conversely, to show every member of A355641 is divisible by at least one of those six numbers, we consider all positive integer solutions of the equation $1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}$ . Without loss of generality, we may assume $a \le b \le c \le d \le e$. In particular, $\frac{5}{a} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1$ so $a \le 5$. Similarly, given $a$, $\frac{4}{b} \ge \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = 1 - \frac{1}{a}$, so $b \le \frac{4}{1 - \frac{1}{a}}$. On the other hand, $\frac{1}{a} + \frac{1}{b} < 1$ so $b > \frac{1}{1 - \frac{1}{a}}$. Similarly we can get upper and lower bounds for each of the variables in terms of the previous ones, and enumerate all solutions $[a, b, c, d, e]$ using the following Maple program.

```
> R:= NULL:
  for a from 2 to 5 do
    x:= 1-1/a;
    for b from max(a,ceil(1/x)) to floor(4/x) do
       y:= x-1/b; if y = 0 then next fi;
       for c from max(b,ceil(1/y)) to floor(3/y) do
```

```
            z:= y-1/c; if z = 0 then next fi;
            for d from max(c,ceil(1/z)) to floor(2/z) do
              w:= z-1/d; if w = 0 then next fi;
              e:= 1/w; if e::integer and e >= d then R:= R, [a,b,c,d,
  e] fi
  od od od od:
```

$nops([R])$

$$147 \tag{1}$$

It turns out there are 147 solutions with $a \le b \le c \le d \le e$. Each corresponds to a possible pattern $k = \frac{k}{a} + \frac{k}{b} + \frac{k}{c} + \frac{k}{d} + \frac{k}{e}$ for writing $k$ as the sum of 5 of its divisors which works when $k$ is divisible by each of $a, b, c, d, e$. We look for solutions that do not have the property that at least one of $a, b, c, d, e$ is divisible by at least one of 5, 6, 8, 9, 14 and 21.

```
> remove(t -> member(0, t mod 5) or member(0, t mod 6) or member(0,
  t mod 8) or member(0, t mod 9) or member(0, t mod 14) or member
  (0, t mod 21), [R]);
```

$$[[2, 3, 11, 22, 33]] \tag{2}$$

There is one such solution, but in order for this to work for $k$, $k$ must be divisible by 2 and 3 and thus by 6.

For those who wish to check out the full list of solutions, here they are:

$R$

[2, 3, 7, 43, 1806], [2, 3, 7, 44, 924], [2, 3, 7, 45, 630], [2, 3, 7, 46, 483], [2, 3, 7, 48, 336], [2, 3, 7, 49, 294], [2, 3, 7, 51, 238], [2, 3, 7, 54, 189], [2, 3, 7, 56, 168], [2, 3, 7, 60, 140], [2, 3, 7, 63, 126], [2, 3, 7, 70, 105], [2, 3, 7, 78, 91], [2, 3, 7, 84, 84], [2, 3, 8, 25, 600], [2, 3, 8, 26, 312], [2, 3, 8, 27, 216], [2, 3, 8, 28, 168], [2, 3, 8, 30, 120], [2, 3, 8, 32, 96], [2, 3, 8, 33, 88], [2, 3, 8, 36, 72], [2, 3, 8, 40, 60], [2, 3, 8, 42, 56], [2, 3, 8, 48, 48], [2, 3, 9, 19, 342], [2, 3, 9, 20, 180], [2, 3, 9, 21, 126], [2, 3, 9, 22, 99], [2, 3, 9, 24, 72], [2, 3, 9, 27, 54], [2, 3, 9, 30, 45], [2, 3, 9, 36, 36], [2, 3, 10, 16, 240], [2, 3, 10, 18, 90], [2, 3, 10, 20, 60], [2, 3, 10, 24, 40], [2, 3, 10, 30, 30], [2, 3, 11, 14, 231], [2, 3, 11, 15, 110], [2, 3, 11, 22, 33], [2, 3, 12, 13, 156], [2, 3, 12, 14, 84], [2, 3, 12, 15, 60], [2, 3, 12, 16, 48], [2, 3, 12, 18, 36], [2, 3, 12, 20, 30], [2, 3, 12, 21, 28], [2, 3, 12, 24, 24], [2, 3, 13, 13, 78], [2, 3, 14, 14, 42], [2, 3, 14, 15, 35], [2, 3, 14, 21, 21], [2, 3, 15, 15, 30], [2, 3, 15, 20, 20], [2, 3, 16, 16, 24], [2, 3, 18, 18, 18], [2, 4, 5, 21, 420], [2, 4, 5, 22, 220], [2, 4, 5, 24, 120], [2, 4, 5, 25, 100], [2, 4, 5, 28, 70], [2, 4, 5, 30, 60], [2, 4, 5, 36, 45], [2, 4, 5, 40, 40], [2, 4, 6, 13, 156], [2, 4, 6, 14, 84], [2, 4, 6, 15, 60], [2, 4, 6, 16, 48], [2, 4, 6, 18, 36], [2, 4, 6, 20, 30], [2, 4, 6, 21, 28], [2, 4, 6, 24, 24], [2, 4, 7, 10, 140], [2, 4, 7, 12, 42], [2, 4, 7, 14, 28], [2, 4, 8, 9, 72], [2, 4, 8, 10, 40], [2, 4, 8, 12, 24], [2, 4, 8, 16, 16], [2, 4, 9, 9, 36], [2, 4, 9, 12, 18], [2, 4, 10, 10, 20], [2, 4, 10, 12, 15], [2, 4, 12, 12, 12], [2, 5, 5, 11, 110], [2, 5, 5, 12, 60], [2, 5, 5, 14, 35], [2, 5, 5, 15, 30], [2, 5, 5, 20, 20], [2, 5, 6, 8, 120], [2, 5, 6, 9, 45], [2, 5, 6, 10, 30], [2, 5, 6, 12, 20], [2, 5, 6, 15, 15], [2, 5, 7, 7, 70], [2, 5, 8, 8, 20], [2, 5, 10, 10, 10], [2, 6, 6, 7, 42], [2, 6, 6, 8, 24], [2, 6, 6, 9, 18], [2, 6, 6, 10, 15], [2, 6, 6, 12, 12], [2, 6, 7, 7, 21], [2, 6, 8, 8, 12], [2, 6, 9, 9, 9], [2, 7, 7, 7, 14], [2, 8, 8, 8, 8], [3, 3, 4, 13, 156], [3, 3, 4, 14, 84], [3, 3, 4, 15, 60], [3, 3, 4, 16, 48], [3, 3, 4, 18, 36], [3, 3, 4, (3) 20, 30], [3, 3, 4, 21, 28], [3, 3, 4, 24, 24], [3, 3, 5, 8, 120], [3, 3, 5, 9, 45], [3, 3, 5, 10, 30], [3, 3, 5, 12, 20], [3, 3, 5, 15, 15], [3, 3, 6, 7, 42], [3, 3, 6, 8, 24], [3, 3, 6, 9, 18], [3, 3, 6, 10, 15], [3, 3, 6, 12, 12], [3, 3, 7, 7, 21], [3, 3, 8, 8, 12], [3, 3, 9, 9, 9], [3, 4, 4, 7, 42], [3, 4, 4, 8, 24], [3, 4, 4, 9, 18], [3, 4, 4, 10, 15], [3, 4, 4, 12, 12], [3, 4, 5, 5, 60], [3, 4, 5, 6, 20], [3, 4, 6, 6, 12], [3, 4, 6, 8, 8], [3, 5, 5, 5, 15], [3, 5, 5, 6, 10], [3, 6, 6, 6, 6], [4, 4, 4, 5, 20], [4, 4, 4, 6, 12], [4, 4, 4, 8, 8], [4, 4, 5, 5, 10], [4, 4, 6, 6, 6], [5, 5, 5, 5, 5]