Bounds on t_i when the product of factors $(3+\frac{1}{t_i})$ is given

Let $n \in \mathbb{N}$, $0 < t_1 \le \cdots \le t_n$ and

$$p \ := \ \prod_{i=1}^{n} \left(3 + \frac{1}{t_i}\right). \tag{1}$$

Assuming $0 \le m < n$ and $t_1, \ldots, t_m$ specified, we derive bounds for t_{m+1}. Set

$$u_m \ := \ \prod_{i=1}^{m} \left(3 + \frac{1}{t_i}\right) \quad \text{and} \quad v_m \ := \ \prod_{i=m+1}^{n} \left(3 + \frac{1}{t_i}\right), \tag{2}$$

where empty products are defined to be 1, so $u_0 = 1$. Then

$$p \ = \ u_m\, v_m \ \ge \ u_m \left(3 + \frac{1}{t_{m+1}}\right) 3^{n-m-1} \tag{3}$$

gives

$$\frac{1}{t_{m+1}} \ \le \ \frac{p}{u_m\, 3^{n-m-1}} - 3. \tag{4}$$

The right hand side is positive because $p = u_m\, v_m > u_m\, 3^{n-m}$, hence:

$$t_{m+1} \ \ge \ \frac{1}{\frac{p}{u_m\, 3^{n-m-1}} - 3} \tag{5}$$

Furthermore,

$$p \ = \ u_m\, v_m \ \le \ u_m \left(3 + \frac{1}{t_{m+1}}\right)^{n-m} \tag{6}$$

gives

$$\frac{1}{t_{m+1}} \ \ge \ \left(\frac{p}{u_m}\right)^{\frac{1}{n-m}} - 3. \tag{7}$$

Again, the right hand side is positive because $p = u_m\, v_m > u_m\, 3^{n-m}$, hence:

$$t_{m+1} \ \le \ \frac{1}{\left(\frac{p}{u_m}\right)^{\frac{1}{n-m}} - 3} \tag{8}$$

Bounds (5) and (8) were used for calculating terms of A355626 and the related sequences with the PARI program attached to A355626. There, the t_i are integers with $2 \le t_1 \le \cdots \le t_n$. Given an integer target value for the product p, for which obviously $3^n < p < 3.5^n$ must hold, said PARI program loops over the feasible values of $t_1, \ldots, t_{n-1}$, calculates the corresponding value of t_n, and checks whether this value of t_n is integer. For $m < n-1$ we can write $>$ instead of $\ge$ in (3) and in (5), so, t_{m+1} is bounded from below by the smallest integer that is larger than the right hand side of 5.

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