Trunk Generating Functions for n-ominoes of Dimension $n-3$

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Golomb [1, pages 81-84] shows that we can represent multidimensional polyominoes as graphs, each vertex of the graph being the center of a cell, and each edge being the line segment connecting the centers of adjacent cells. The dimension of the graph (and its source polyomino) is that of the convex hull of the cell centers. Golomb notes that the number of unoriented n-ominoes of dimension $n-1$ is equivalent to the number of free trees with n nodes (A000055). For unoriented polyominoes, chiral pairs are counted as one.

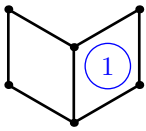
To enumerate n-ominoes of dimension $n-3$, we want to reduce the dimension of each polyomino graph by a removal process in order to obtain a small number of irreducible canonical graphs that we shall call *trunks*. First we align the graph edges with the Cartesian coordinate axes. In the removal process, an edge that is not parallel to any other edge is removed by changing the coordinate along an axis parallel to it to be the same for all vertices. In the first removal process, we remove any vertex and the single edge to which it is attached if that edge is not parallel to another edge of the graph. Our second removal involves a vertex connected to exactly two edges, neither of which is parallel to any other edge. We can eliminate this vertex by removing just one of the two edges. We shall not do this if it causes a new edge to arise elsewhere by bringing two other vertices into proximity. Note that each removal reduces the order of the polyomino by one and also reduces its dimension by one, so that the difference between the order of the polyomino and its dimension is unchanged.

It is not difficult to determine all the trunks that can arise for n-ominoes of dimension $n-3$. For each trunk we can determine a generating function that will indicate the exact number of n-ominoes that reduce to that trunk. There are separate generating functions for unoriented, chiral, and asymmetric n-ominoes. In addition, we can distinguish trunks of orthoplex polyominoes from those that have at least one coordinate with more than two values, which we call *extended* polyominoes. We sum together the generating functions for each trunk to obtain the generating function for all n-ominoes of dimension $n-3$. There are 56 trunks for orthoplex polyominoes and 173 for extended polyominoes. For multidimensional polyominoes we use all 229 trunks.

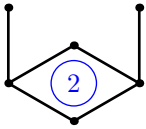
We have done this process for n-ominoes of dimension $n-2$. The results for multidimensional polyominoes are A036364 (unoriented), A036365 (chiral), and A036366 (asymmetric). For orthoplex polyominoes we have A036367 (unoriented), A036368 (chiral), and A036369 (asymmetric).

References

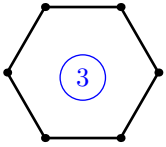
[1] S. W. Golomb. *Polyominoes.* Princeton University Press, second edition, 1994.



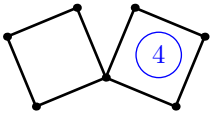
$$\left(A(x)^6 - A(x)^2A(x^2)^2 - 2A(x^2)^3 + 2A(x^2)A(x^4)\right)/4 = x^7 + 3x^8 + 13x^9 + 35x^{10} + 98x^{11} + 245x^{12} + 626x^{13} + 1535x^{14} + 3788x^{15} + 9210x^{16} + 22441x^{17} + 54376x^{18} + 131876x^{19} + 319215x^{20} + \dots$$



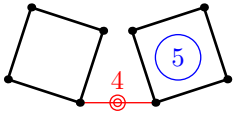
$$\left(A(x)^4 - A(x^2)^2\right)\left(A(x)^2 - A(x^2)\right)/4 = 2x^8 + 6x^9 + 22x^{10} + 60x^{11} + 166x^{12} + 426x^{13} + 1094x^{14} + 2732x^{15} + 6808x^{16} + 16780x^{17} + 41276x^{18} + 101060x^{19} + 247132x^{20} + \dots$$



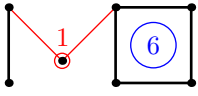
$$\left(A(x)^6 - 3A(x)^2A(x^2)^2 - 4A(x^2)^3 + 6A(x^2)A(x^4) + 2A(x^3)^2 - 2A(x^6)\right)/12 = 3x^9 + 8x^{10} + 26x^{11} + 67x^{12} + 180x^{13} + 448x^{14} + 1133x^{15} + 2778x^{16} + 6850x^{17} + 16703x^{18} + 40782x^{19} + 99148x^{20} + \dots$$



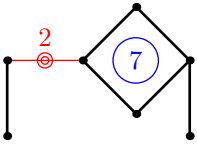
$$A(x)\left(A(x)^6 + A(x)^2A(x^2)^2 - 2A(x^2)\left(A(x)^4 + A(x^2)^2 - A(x^4)\right)\right)/8 = 2x^{10} + 8x^{11} + 29x^{12} + 87x^{13} + 251x^{14} + 684x^{15} + 1825x^{16} + 4755x^{17} + 12242x^{18} + 31155x^{19} + 78734x^{20} + \dots$$



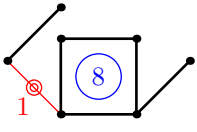
$$A(x)^4\left(A(x)^2 - A(x^2)\right)^2/(8(1-A(x))) - (1+A(x))A(x^2)^2\left(A(x^2)^2 - A(x^4)\right)/(4(1-A(x^2))) = 3x^{11} + 14x^{12} + 62x^{13} + 215x^{14} + 715x^{15} + 2191x^{16} + 6531x^{17} + 18759x^{18} + 52900x^{19} + 146233x^{20} + \dots$$



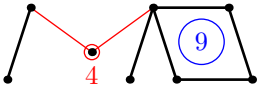
$$A(x)^7/(1-A(x)) = x^7 + 8x^8 + 37x^9 + 137x^{10} + 450x^{11} + 1371x^{12} + 3980x^{13} + 11182x^{14} + 30689x^{15} + 82786x^{16} + 220438x^{17} + 581109x^{18} + 1519849x^{19} + 3950057x^{20} + \dots$$



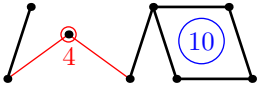
$$A(x)^5\left(A(x)^2 - A(x^2)\right)/(2(1-A(x))) = x^8 + 7x^9 + 31x^{10} + 114x^{11} + 375x^{12} + 1150x^{13} + 3368x^{14} + 9550x^{15} + 26447x^{16} + 71957x^{17} + 193142x^{18} + 512934x^{19} + 1350724x^{20} + \dots$$



$$A(x)^7/(1-A(x)) = x^7 + 8x^8 + 37x^9 + 137x^{10} + 450x^{11} + 1371x^{12} + 3980x^{13} + 11182x^{14} + 30689x^{15} + 82786x^{16} + 220438x^{17} + 581109x^{18} + 1519849x^{19} + 3950057x^{20} + \dots$$



$$A(x)^6\left(A(x)^2 - A(x^2)\right)/(2(1-A(x))) = x^9 + 8x^{10} + 39x^{11} + 154x^{12} + 537x^{13} + 1728x^{14} + 5268x^{15} + 15455x^{16} + 44073x^{17} + 123010x^{18} + 337650x^{19} + 914668x^{20} + \dots$$



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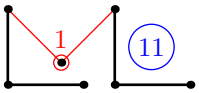
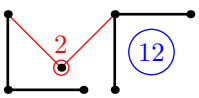
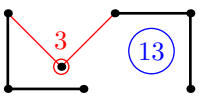


$$A(x)^7/(2(1-A(x))) - (A(x)+A(x^2))A(x^2)^3/(2(1-A(x^2))) = 3x^8 + 16x^9 + 63x^{10} + 216x^{11} + 665x^{12} + 1958x^{13} + 5518x^{14} + 15227x^{15} + 41126x^{16} + 109755x^{17} + 289505x^{18} + 757934x^{19} + 1970526x^{20} + \dots$$



$$A(x)^7/(1-A(x)) = x^7 + 8x^8 + 37x^9 + 137x^{10} + 450x^{11} + 1371x^{12} + 3980x^{13} + 11182x^{14} + 30689x^{15} + 82786x^{16} + 220438x^{17} + 581109x^{18} + 1519849x^{19} + 3950057x^{20} + \dots$$



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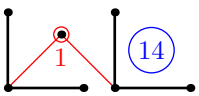
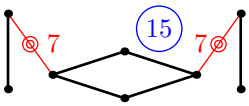
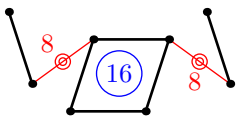


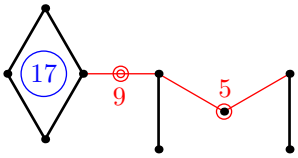
$$A(x)^3\left(A(x)^4 - A(x^2)^2\right)/(4(1-A(x))) - \left(A(x)+A(x^2)\right)\left(A(x^2)^3 - A(x^2)A(x^4)\right)/(2(1-A(x^2))) = x^8 + 5x^9 + 23x^{10} + 84x^{11} + 271x^{12} + 822x^{13} + 2368x^{14} + 6644x^{15} + 18169x^{16} + 48983x^{17} + 130228x^{18} + 343220x^{19} + 897164x^{20} + \dots$$



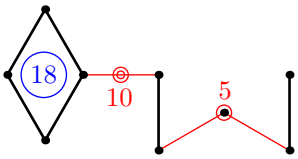
$$\left(A(x)^2 - A(x^2)\right)\left(A(x)^6/(1-A(x))^2 - A(x^2)^3/(1-A(x^2))\right)/4 = 4x^{10} + 21x^{11} + 102x^{12} + 388x^{13} + 1384x^{14} + 4544x^{15} + 14346x^{16} + 43541x^{17} + 128919x^{18} + 373231x^{19} + 1062971x^{20} + \dots$$



$$\left(A(x)^8/(1-A(x))^2 - A(x^2)^4/(1-A(x^2))\right)/2 = 5x^9 + 26x^{10} + 126x^{11} + 474x^{12} + 1679x^{13} + 5462x^{14} + 17113x^{15} + 51531x^{16} + 151508x^{17} + 435664x^{18} + 1233120x^{19} + 3441850x^{20} + \dots$$



$$\left(A(x)^2 - A(x^2)\right)A(x)^7/(2(1-A(x))^2) = x^{10} + 10x^{11} + 59x^{12} + 273x^{13} + 1093x^{14} + 3975x^{15} + 13513x^{16} + 43718x^{17} + 136217x^{18} + 412157x^{19} + 1218273x^{20} + \dots$$



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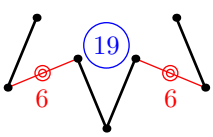
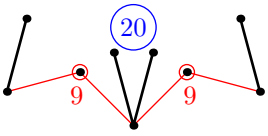
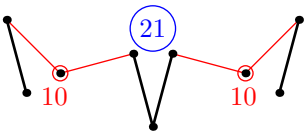


$$A(x)\left(A(x)^6/(1-A(x))^2 - A(x^2)^3/(1-A(x^2))\right)/2 = 4x^8 + 21x^9 + 94x^{10} + 342x^{11} + 1155x^{12} + 3637x^{13} + 11026x^{14} + 32340x^{15} + 92781x^{16} + 261263x^{17} + 725492x^{18} + 1991054x^{19} + 5413204x^{20} + \dots$$



$$A(x)\left(A(x)^8/(1-A(x))^2 - A(x^2)^4/(1-A(x^2))\right)/2 = 5x^{10} + 31x^{11} + 157x^{12} + 636x^{13} + 2346x^{14} + 7975x^{15} + 25796x^{16} + 80079x^{17} + 241379x^{18} + 710144x^{19} + 2050188x^{20} + \dots$$



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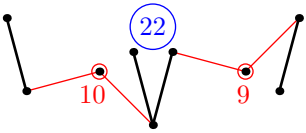
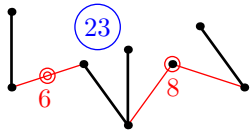
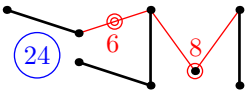


$$A(x)^9/(1-A(x))^2 = x^9 + 11x^{10} + 68x^{11} + 321x^{12} + 1296x^{13} + 4724x^{14} + 16039x^{15} + 51725x^{16} + 160488x^{17} + 483308x^{18} + 1421572x^{19} + 4102736x^{20} + \dots$$



$$A(x)^8/(1-A(x))^2 = x^8 + 10x^9 + 57x^{10} + 252x^{11} + 964x^{12} + 3358x^{13} + 10970x^{14} + 34226x^{15} + 103186x^{16} + 303016x^{17} + 871651x^{18} + 2466240x^{19} + 6884528x^{20} + \dots$$



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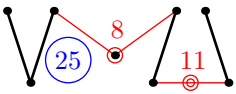

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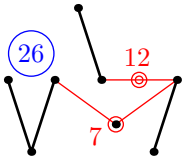

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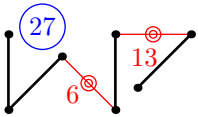
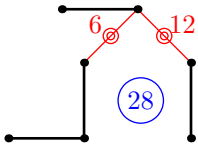


$$A(x)^7/(1-A(x))^2 = x^7 + 9x^8 + 47x^9 + 194x^{10} + 702x^{11} + 2335x^{12} + 7338x^{13} + 22152x^{14} + 64915x^{15} + 185972x^{16} + 523454x^{17} + 1452760x^{18} + 3986089x^{19} + 10834585x^{20} + \dots$$



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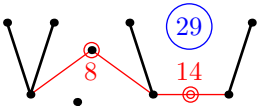
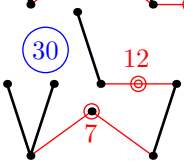
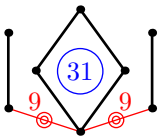


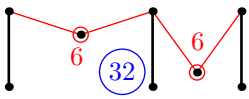
$$A(x)^8/(1-A(x))^2 = x^8 + 10x^9 + 57x^{10} + 252x^{11} + 964x^{12} + 3358x^{13} + 10970x^{14} + 34226x^{15} + 103186x^{16} + 303016x^{17} + 871651x^{18} + 2466240x^{19} + 6884528x^{20} + \dots$$



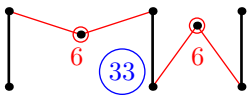
$$A(x)^8/(1-A(x))^2 = x^8 + 10x^9 + 57x^{10} + 252x^{11} + 964x^{12} + 3358x^{13} + 10970x^{14} + 34226x^{15} + 103186x^{16} + 303016x^{17} + 871651x^{18} + 2466240x^{19} + 6884528x^{20} + \dots$$



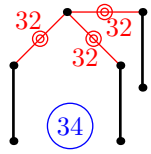
$$A(x)^2\left(A(x)^2-A(x^2)\right)\left(A(x)^4/(1-A(x))^2-A(x^2)^2/(1-A(x^2))\right)/4$$
$$=3x^{10}+19x^{11}+92x^{12}+366x^{13}+1315x^{14}+4386x^{15}+13924x^{16}$$
$$+42559x^{17}+126467x^{18}+367475x^{19}+1049009x^{20}+\dots$$



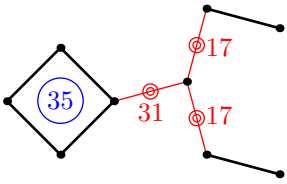
$$A(x)^2\left(A(x)^6/(1-A(x))^2-A(x^2)^3/(1-A(x^2))\right)/2=4x^9$$
$$+25x^{10}+119x^{11}+465x^{12}+1645x^{13}+5409x^{14}+16958x^{15}$$
$$+51255x^{16}+150799x^{17}+434275x^{18}+1229775x^{19}$$
$$+3434851x^{20}+\dots$$



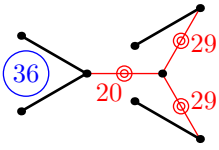
$$\left(A(x)^8/(1-A(x))^2-A(x^2)^4/(1-A(x^2))\right)/2=5x^9+26x^{10}$$
$$+126x^{11}+474x^{12}+1679x^{13}+5462x^{14}+17113x^{15}+51531x^{16}$$
$$+151508x^{17}+435664x^{18}+1233120x^{19}+3441850x^{20}+\dots$$



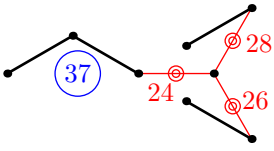
$$\left(A(x)^7/(1-A(x))^3-3A(x)^3A(x^2)^2/\left((1-A(x))(1-A(x^2)\right)\right.$$
$$\left.+2A(x)A(x^3)^2/(1-A(x^3))\right)/6=3x^9+25x^{10}+118x^{11}$$
$$+474x^{12}+1712x^{13}+5752x^{14}+18409x^{15}+56857x^{16}$$
$$+170807x^{17}+502253x^{18}+1451589x^{19}+4136449x^{20}+\dots$$



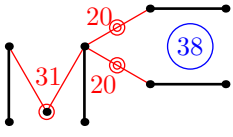
$$A(x)^3(A(x)^2-A(x^2))\left(A(x)^4/(1-A(x))^2-A(x^2)^2/(1-A(x^2))\right)$$
$$/(4(1-A(x)))=3x^{11}+25x^{12}+142x^{13}+653x^{14}+2646x^{15}$$
$$+9826x^{16}+34285x^{17}+114127x^{18}+366314x^{19}+1142135x^{20}+\dots$$



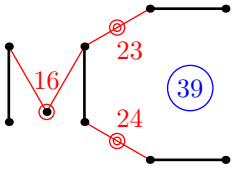
$$A(x)^2\left(A(x)^6/(1-A(x))^2-A(x^2)^3/(1-A(x^2))\right)/(2(1-A(x)))$$
$$=4x^9+29x^{10}+152x^{11}+650x^{12}+2484x^{13}+8765x^{14}$$
$$+29272x^{15}+93754x^{16}+290868x^{17}+879808x^{18}$$
$$+2607508x^{19}+7599176x^{20}+\dots$$



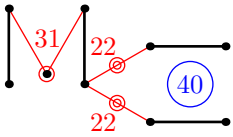
$$A(x)^8/(1-A(x))^3=x^8+11x^9+69x^{10}+333x^{11}+1379x^{12}$$
$$+5164x^{13}+18023x^{14}+59743x^{15}+190444x^{16}+588844x^{17}$$
$$+1776900x^{18}+5256918x^{19}+15300266x^{20}+\dots$$



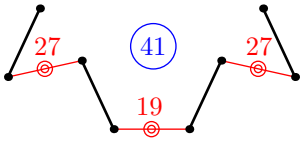
$$A(x)^5\left(A(x)^4/(1-A(x))^2-A(x^2)^2/(1-A(x^2))\right)/(2(1-A(x)))$$
$$=3x^{10}+28x^{11}+164x^{12}+767x^{13}+3129x^{14}+11646x^{15}$$
$$+40608x^{16}+134882x^{17}+431593x^{18}+1340876x^{19}$$
$$+4067990x^{20}+\dots$$



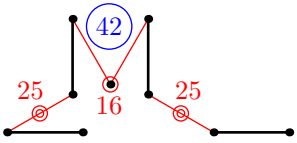
$$A(x)^9/(1-A(x))^3=x^9+12x^{10}+81x^{11}+415x^{12}+1806x^{13}$$
$$+7053x^{14}+25517x^{15}+87258x^{16}+285828x^{17}+905249x^{18}$$
$$+2790678x^{19}+8415738x^{20}+\dots$$



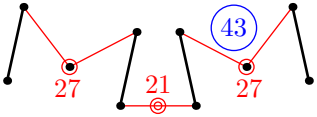
$$A(x)^5\left(A(x)^4/(1-A(x))^2-A(x^2)^2/(1-A(x^2))\right)/(2(1-A(x)))$$
$$=3x^{10}+28x^{11}+164x^{12}+767x^{13}+3129x^{14}+11646x^{15}$$
$$+40608x^{16}+134882x^{17}+431593x^{18}+1340876x^{19}$$
$$+4067990x^{20}+\dots$$



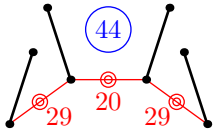
$$\left(A(x)^8/(1-A(x))^3-(1+A(x))A(x^2)^4/(1-A(x^2))^2\right)/2$$
$$= 5x^9+31x^{10}+163x^{11}+674x^{12}+2566x^{13}+8953x^{14}$$
$$+29807x^{15}+95015x^{16}+294172x^{17}+887724x^{18}$$
$$+2627478x^{19}+7647503x^{20}+\dots$$



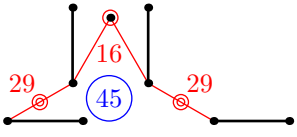
$$\left(A(x)^9/(1-A(x))^3-(A(x)+A(x^2))A(x^2)^4/(1-A(x^2))^2\right)/2$$
$$= 5x^{10}+37x^{11}+200x^{12}+887x^{13}+3491x^{14}+12694x^{15}$$
$$+43484x^{16}+142664x^{17}+452060x^{18}+1394358x^{19}$$
$$+4205653x^{20}+\dots$$



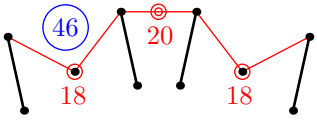
$$\left(A(x)^{10}/(1-A(x))^3-(1+A(x))A(x^2)^5/(1-A(x^2))^2\right)/2$$
$$= 6x^{11}+43x^{12}+251x^{13}+1145x^{14}+4719x^{15}+17688x^{16}$$
$$+62585x^{17}+210681x^{18}+684214x^{19}+2155465x^{20}+\dots$$



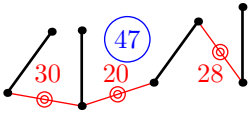
$$\left(A(x)^8/(1-A(x))^3-(1+A(x))A(x^2)^4/(1-A(x^2))^2\right)/2$$
$$= 5x^9+31x^{10}+163x^{11}+674x^{12}+2566x^{13}+8953x^{14}$$
$$+29807x^{15}+95015x^{16}+294172x^{17}+887724x^{18}$$
$$+2627478x^{19}+7647503x^{20}+\dots$$



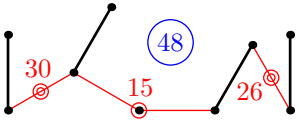
$$\left(A(x)^9/(1-A(x))^3-(A(x)+A(x^2))A(x^2)^4/(1-A(x^2))^2\right)/2$$
$$= 5x^{10}+37x^{11}+200x^{12}+887x^{13}+3491x^{14}+12694x^{15}$$
$$+43484x^{16}+142664x^{17}+452060x^{18}+1394358x^{19}$$
$$+4205653x^{20}+\dots$$



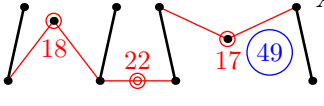
$$\left(A(x)^{10}/(1-A(x))^3-(1+A(x))A(x^2)^5/(1-A(x^2))^2\right)/2$$
$$= 6x^{11}+43x^{12}+251x^{13}+1145x^{14}+4719x^{15}+17688x^{16}$$
$$+62585x^{17}+210681x^{18}+684214x^{19}+2155465x^{20}+\dots$$



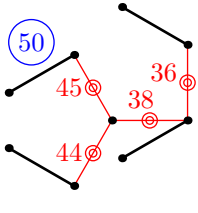
$$A(x)^8/(1-A(x))^3=x^8+11x^9+69x^{10}+333x^{11}+1379x^{12}$$
$$+5164x^{13}+18023x^{14}+59743x^{15}+190444x^{16}+588844x^{17}$$
$$+1776900x^{18}+5256918x^{19}+15300266x^{20}+\dots$$



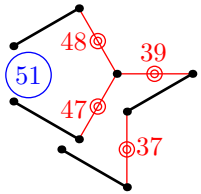
$$A(x)^9/(1-A(x))^3=x^9+12x^{10}+81x^{11}+415x^{12}+1806x^{13}$$
$$+7053x^{14}+25517x^{15}+87258x^{16}+285828x^{17}+905249x^{18}$$
$$+2790678x^{19}+8415738x^{20}+\dots$$



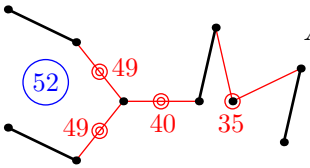
$$A(x)^{10}/(1-A(x))^3=x^{10}+13x^{11}+94x^{12}+510x^{13}$$
$$+2329x^{14}+9478x^{15}+35533x^{16}+125340x^{17}$$
$$+421941x^{18}+1369106x^{19}+4313002x^{20}+\dots$$



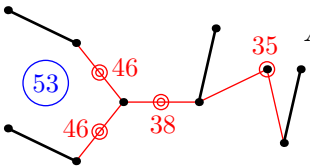
$$A(x)^9/(1-A(x))^4=x^9+13x^{10}+95x^{11}+524x^{12}+2440x^{13}$$
$$+10141x^{14}+38857x^{15}+140119x^{16}+482148x^{17}+1598636x^{18}$$
$$+5143810x^{19}+16147023x^{20}+\dots$$



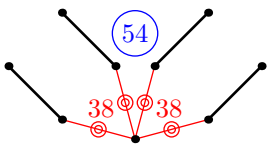
$$A(x)^9/(1-A(x))^4 = x^9 + 13x^{10} + 95x^{11} + 524x^{12} + 2440x^{13} + 10141x^{14} + 38857x^{15} + 140119x^{16} + 482148x^{17} + 1598636x^{18} + 5143810x^{19} + 16147023x^{20} + \ldots$$



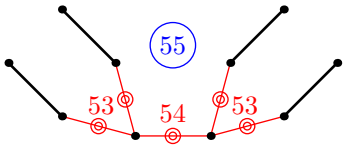
$$A(x)^6\left(A(x)^4/(1-A(x))^2 - A(x^2)^2/(1-A(x^2))\right)/(2(1-A(x))^2) = 3x^{11} + 34x^{12} + 232x^{13} + 1234x^{14} + 5631x^{15} + 23146x^{16} + 88208x^{17} + 317479x^{18} + 1092860x^{19} + 3630304x^{20} + \ldots$$



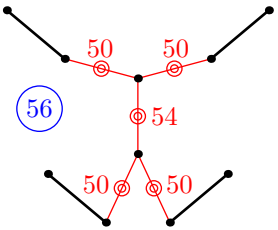
$$A(x)^6\left(A(x)^4/(1-A(x))^2 - A(x^2)^2/(1-A(x^2))\right)/(2(1-A(x))^2) = 3x^{11} + 34x^{12} + 232x^{13} + 1234x^{14} + 5631x^{15} + 23146x^{16} + 88208x^{17} + 317479x^{18} + 1092860x^{19} + 3630304x^{20} + \ldots$$



$$A(x)\left(A(x)^4/(1-A(x))^2 - A(x^2)^2/(1-A(x^2))\right)^2/8 - A(x)\left(A(x^2)^4/(1-A(x^2))^2 - A(x^4)^2/(1-A(x^4))\right)/4 = 3x^{11} + 33x^{12} + 192x^{13} + 911x^{14} + 3769^{15} + 14347x^{16} + 51231x^{17} + 174643x^{18} + 573719x^{19} + 1830505x^{20} + \ldots$$



$$A(x)^2\left(A(x)^4/(1-A(x))^2 - A(x^2)^2/(1-A(x^2))\right)^2/(8(1-A(x))) - A(x^2)(1+A(x))\left(A(x^2)^4/(1-A(x^2))^2 - A(x^4)^2/(1-A(x^4))\right)/(4(1-A(x^2))) = 3x^{12} + 42x^{13} + 276x^{14} + 1485x^{15} + 6784x^{16} + 28292x^{17} + 109421x^{18} + 401332x^{19} + 1408698x^{20} + \ldots$$



$$A(x)^2\left(A(x)^8/(1-A(x))^4 - A(x^2)^4/(1-A(x^2))^2\right)/(4(1-A(x))) - A(x^2)^5(1+A(x))/(2(1-A(x^2))^3) + A(x^2)A(x^4)^2(1+A(x))/(2(1-A(x^4))(1-A(x^2))) = 3x^{11} + 25x^{12} + 182x^{13} + 956x^{14} + 4484x^{15} + 18787x^{16} + 73684x^{17} + 272542x^{18} + 966282x^{19} + 3303587x^{20} + \ldots$$