Possible numbers of divisors of terms in A354715

For $n \in \mathbb{N}^*$, let $d(n)$ be the number of divisors of n. Write

$$S = \{d(k) : d(k) \mid k - 2\}.$$

Proposition. *(i) Every even number is in S;*

(ii) Let m be an odd number, $p_1, p_2, \cdots, p_r$ be the distinct prime factors of m, then $m \in S$ if and only if

$$\gcd(p_1 - 1, \cdots, p_r - 1) = \gcd\left(\frac{p_1 - 1}{\mathrm{ord}_{p_1}(2)}, \cdots, \frac{p_i - 1}{\mathrm{ord}_{p_i}(2)}\right),$$

where $\mathrm{ord}_m(a)$ is the multiplicative order of a modulo m.

Proof. (i) Obviously $2 \in S$. For even $m > 2$, write $m = 2t$. Pick a prime $p \equiv 1 (\mathrm{mod}\ t)$, then $d(2p^{t-1}) = d(2)d(p^{t-1}) = 2t = m$, and we have $2p^{t-1} \equiv 2 (\mathrm{mod}\ m)$.

(ii) We first show that $m \in S \Leftrightarrow x^{\gcd(p_1-1,\cdots,p_r-1)} \equiv 2 (\mathrm{mod}\ m)$ has solutions.

$\Rightarrow$: Suppose that $m \in S$. Let k be such that $m = d(k) \mid k - 2$, write $k = q_1^{m_1-1} \cdots q_k^{m_k-1}$, where q_j's are distinct primes, $m_j > 1$, $m_1 m_2 \cdots m_k = m$. Since m_j's are products of p_i's (the prime factors of m_j's are not necessarily distinct), $p_i \equiv 1 (\mathrm{mod}\ d) (1 \le i \le r)$ implies $m_j \equiv 1 (\mathrm{mod}\ d) (1 \le j \le k)$, namely we have

$$\gcd(p_1 - 1, \cdots, p_r - 1) \mid \gcd(m_1 - 1, \cdots, m_k - 1).$$

Hence 2 is a $\gcd(p_1 - 1, \cdots, p_r - 1)-$th power modulo m.

$\Leftarrow$: Suppose that $x^{\gcd(p_1-1,\cdots,p_r-1)} \equiv 2 (\mathrm{mod}\ m)$. By Bezout's theorem, there exists λ_i, $1 \le i \le r$ such that $\sum_{i=1}^{r} \lambda_i (p_i - 1) = \gcd(p_1 - 1, \cdots, p_r - 1)$. Let $q_1, \cdots, q_r, q$ be distinct primes such that

$$q_i \equiv x^{\lambda_i} (\mathrm{mod}\ m), q \equiv 1 (\mathrm{mod}\ m)$$

(this is possible since x and m are coprime). Write

$$k = q_1^{p_1-1} \cdots q_r^{p_r-1} q^{\frac{m}{p_1 p_2 \cdots p_r} - 1},$$

then $d(k) = m$, and $k \equiv \prod_{i=1}^{r} (x^{\lambda_i})^{p_i-1} \equiv 2 (\mathrm{mod}\ m)$.

(ii) follows by noting that

$$\begin{aligned}
&x^{\gcd(p_1-1,\cdots,p_r-1)} \equiv 2 (\mathrm{mod}\ m) \text{ has solutions} \\
\Leftrightarrow &x^{\gcd(p_1-1,\cdots,p_r-1)} \equiv 2 (\mathrm{mod}\ p_i^{e_i}) \text{ has solutions}, 1 \le i \le r (\text{where } m = \prod_{i=1}^{r} p_i^{e_i}) \\
\Leftrightarrow &x^{\gcd(p_1-1,\cdots,p_r-1)} \equiv 2 (\mathrm{mod}\ p_i) \text{ has solutions}, 1 \le i \le r (\text{since } p_i\text{'s are odd primes}) \\
\Leftrightarrow &\mathrm{ord}_{p_i}(2) \mid \frac{p_i - 1}{\gcd(p_1 - 1, \cdots, p_r - 1)}, 1 \le i \le r \\
\Leftrightarrow &\gcd(p_1 - 1, \cdots, p_r - 1) \mid \gcd\left(\frac{p_1 - 1}{\mathrm{ord}_{p_1}(2)}, \cdots, \frac{p_i - 1}{\mathrm{ord}_{p_i}(2)}\right) \\
\Leftrightarrow &\gcd(p_1 - 1, \cdots, p_r - 1) = \gcd\left(\frac{p_1 - 1}{\mathrm{ord}_{p_1}(2)}, \cdots, \frac{p_i - 1}{\mathrm{ord}_{p_i}(2)}\right).
\end{aligned}$$

□