Width 2 polyominoes.

The sequence A335711(n) counts width 2 polyominoes of height n.

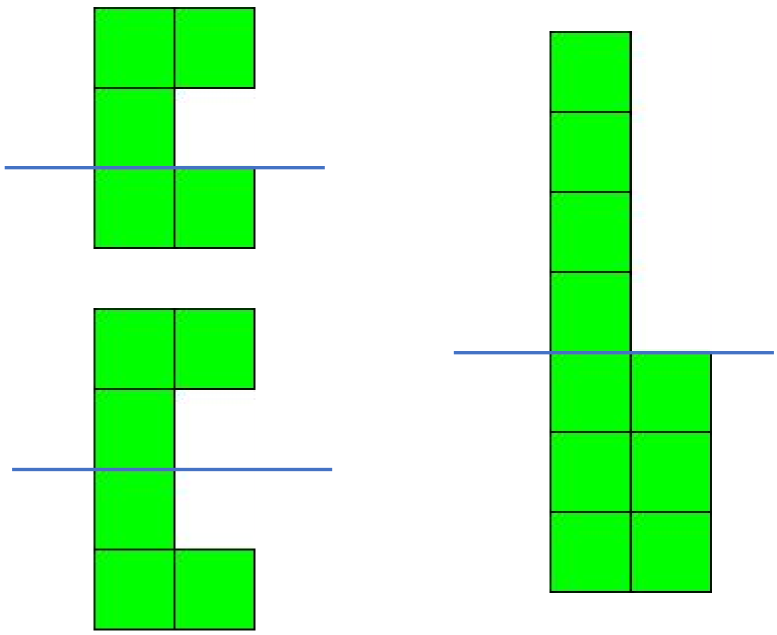
This document has the scope of counting width 2 polyominoes of size n cells.

This table enumerates such polyominoes.

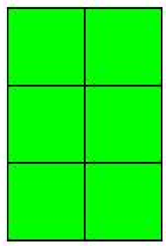
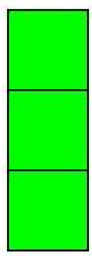
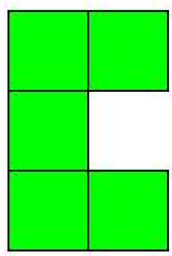
1	0
2	1
3	1
4	4
5	5
6	12
7	18
8	37
9	60
10	117
11	200
12	379
13	669
14	1250
15	2247
16	4168
17	7570
18	13987
19	25549
20	47108
21	86319
22	158978
23	291806
24	537105
25	986786
26	1815699
27	3337560
28	6140047
29	11289571
30	20767180

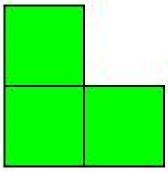
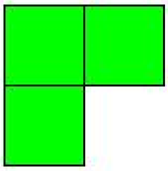
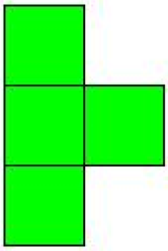
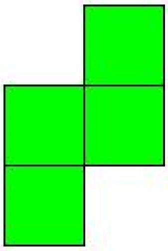
The process of counting the polyominoes is based on recursive formulae, and the idea that any polyomino of width 2 can be divided into 2 polyominoes of (almost) equal size by cutting it into 2 by a horizontal line.

Examples are shown:



Every “half” of a width 2 polyomino can be classified as follows:

Type	Description	Example
a	A complete 2 by m rectangle	
b	A vertical strip, of length m, either on the left or on the right	
d or d’ for polyominoes without or with top to bottom reflective symmetry	An incomplete polyomino with 2 adjacent cells both at the top and at the bottom	

e	A polyomino with just 1 cell at the top and 2 at the bottom	
f. Obviously f and e are equivalent, and in the program, f is suppressed.	A polyomino with 2 cells at the top and just 1 at the bottom	
g or g’ for polyominoes without or with top to bottom reflective symmetry	A polyomino of width 2 with 1 cell at the top, and 1 cell at the bottom on the same side	
h or h’ for polyominoes without or with 180 degree rotational symmetry	A polyomino of width 2 with 1 cell at the top, and 1 cell at the bottom on the other side	

This table shows the number of polyominoes according to their size and classification, counting once left-right reflections, but twice top-bottom reflections.

size	d	d'	e	f	g	g'	h	h'
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	1	1	0	0	0	0
4	0	0	1	1	0	1	0	1
5	0	1	2	2	2	0	2	0
6	0	1	4	4	2	2	2	2
7	2	1	7	7	6	2	8	0
8	2	3	13	13	10	5	12	3
9	8	2	24	24	24	4	28	0
10	12	6	44	44	42	10	46	6
11	30	4	81	81	88	8	96	0
12	50	12	149	149	158	19	166	11
13	108	7	274	274	310	16	326	0
14	188	23	504	504	564	36	580	20
15	376	13	927	927	1074	30	1104	0
16	672	43	1705	1705	1964	67	1994	37
17	1292	24	3136	3136	3680	56	3736	0
18	2340	80	5768	5768	6748	124	6804	68

The formulae for enumerating these polyominoes vary according to the value of n, their size.

In particular, there are 3 different cases:

1. n is even, equal to 2m, for odd m
2. n is even, equal to 2m, for even m
3. n is odd, equal to 2m+1

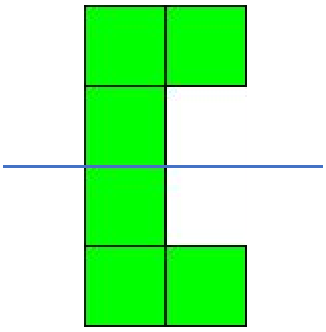
1. n is even, equal to 2m, for odd m

In this case, the width 2 polyomino can be divided in one of the following ways:

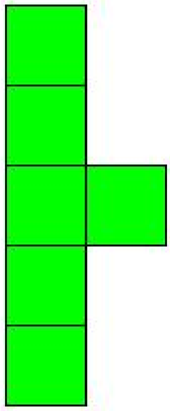
i. a size m polyomino both above and below the dividing line
ii. a size m-1 polyomino above a horizontal domino and the same below

Examples follow.

i:



ii:



1. n is even, equal to 2m, for odd m, sub-case i

Above and below the line there may be polyominoes of each of the classifications a, b, d, d’, e, f, g, g’, h, h’ described above. However, “a” can be excluded as m is odd.

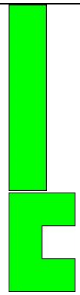
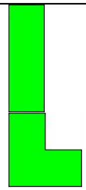
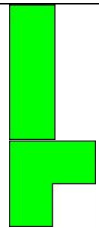
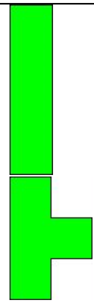
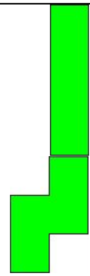
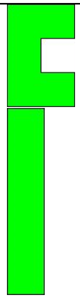
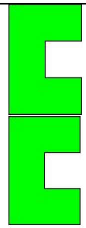
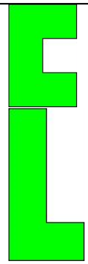
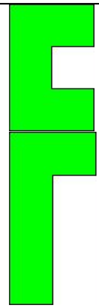
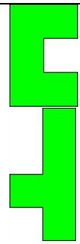
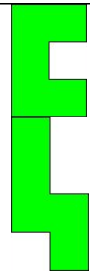
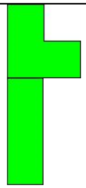
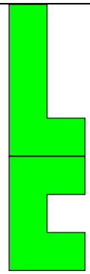
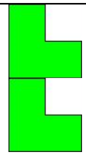
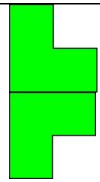
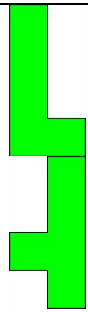
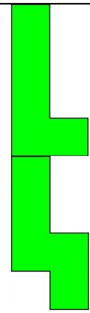
We will now enumerate polyominoes of the classifications d, d’, e, f, g, g’, h, h’ of size n according to the combinations above and below the line, and the numbers of polyominoes of these classifications of size m.

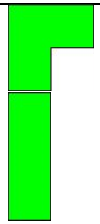
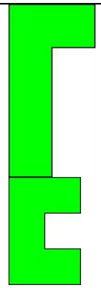
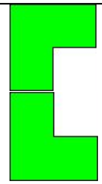
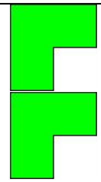
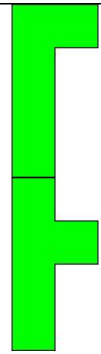
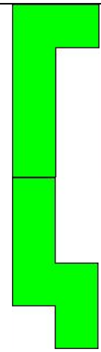
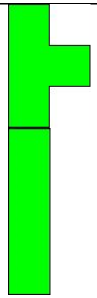
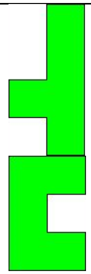
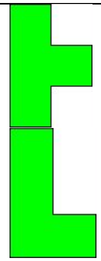
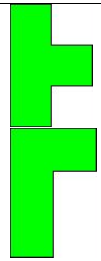
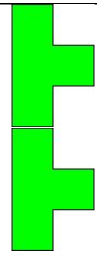
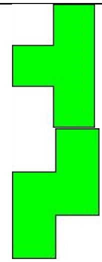
We will consider to be equivalent (i.e., counting as 1) two polyominoes that map on to each other with left-to-right reflection.

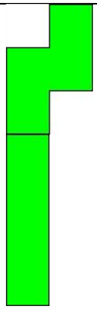
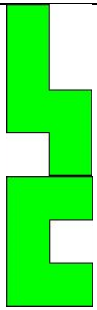
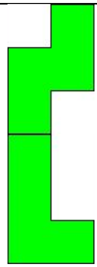
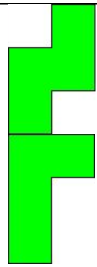
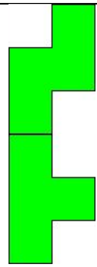
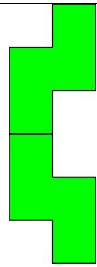
In the formulae below, D(m) is intended to mean d(m)+d’(m), and idem for G and H.

Note that e(i) = f(i) for any i.

For each combination of classification above and below the line, the table below shows an example, and also formulas that indicate the values of each classification of size m contribute to which classification of size n.

	BELOW											
ABOVE	b		d/d’		e		f		g/g’		h/h’	
b	N.a.			e:= 2*D(m)		e:= e(m)		g:= f(m) h:= f(m)		g:= G(m)		h:= H(m)
d/d’		f:= 2*D(m)		d’:= 2*D(m) d:= 2*D(m)*(D(m) -1)		d:= 2*D(m)*e(m)		f:= 2*D(m)*f(m)		f:= 2*D(m)*G(m)		f:= 2*D(m)*H(m)
e		g:= e(m) h:= e(m)		e:= 2*e(m)*D(m)		e:= 2*e(m)*e(m)		g’:= e(m) h’:= e(m) g:= e(m)*(e(m) -1) h:= e(m)*(e(m)-1)		g:= e(m)*G(m) h:= e(m)*G(m)		g:= e(m)*H(m) h:= e(m)*H(m)

	b		d/d’		e		f		g/g’		h/h’	
f		f:= f(m)		d:= 2*f(m)*D(m)		d’:= e(m) d:= e(m)*(f(m)- 1)		f:= 2*f(m)*f(m)		f:= f(m)*G(m)		f:= f(m)*H(m)
g/g’		g:= G(m)		e:= 2*G(m)*D(m)		e:= G(m)*e(m)		g:= G(m)*f(m) h:= G(m)*f(m)		g:= G(m)*(G(m)- 1) g’:= G(m)		h:= G(m)*H(m)

h/h’		h:= H(m)		e:= 2*H(m)*D(m)		e:= H(m)*e(m)		g:= H(m)*f(m) h:= H(m)*f(m)		h:= H(m)*G(m)		g’:= H(m) g:= H(m)*(H(m)-1)

In summary:

d(n)= 2*D(m)*(D(m)-1)   +2*D(m)*e(m)   +2*f(m)*D(m)   +e(m)*(f(m)-1)

d’(n)= 2*D(m) + e(m)

e(n)= 2*D(m) +2*e(m)*D(m)   +2*e(m)*e(m)   +2*G(m)*D(m)   +2*H(m)*D(m)   +e(m)   +G(m)*e(m)   +H(m)*e(m)

f(n)= 2*D(m)   +2*D(m)*f(m) +2*D(m)*G(m)   +2*D(m)*H(m) +2*f(m)*f(m)   +f(m)   +f(m)*G(m)   +f(m)*H(m)

g(n)= e(m) +e(m)*(e(m) -1) +e(m)*G(m) +e(m)*H(m) +f(m) +G(m) +G(m)   +G(m)*(G(m)-1) +G(m)*f(m) +H(m)*(H(m)-1) +H(m)*f(m)

g’(n)= e(m) + G(m)   + H(m)

h(n)= e(m)   +e(m)*(e(m)-1)   +e(m)*G(m)   +e(m)*H(m) +f(m)   +G(m)*f(m) +G(m)*H(m) +H(m) +H(m)   +H(m)*f(m)   +H(m)*G(m)

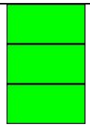
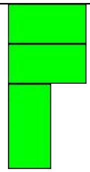
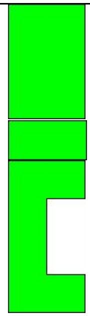
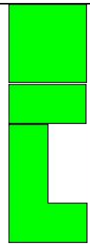
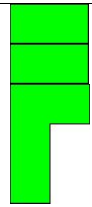
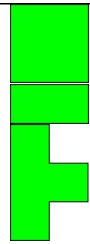
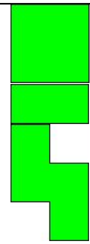
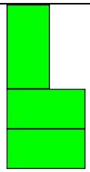
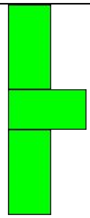
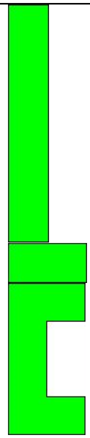
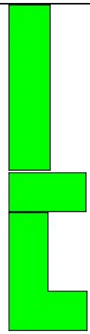
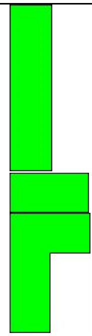
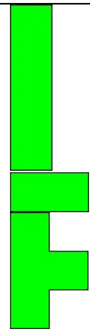
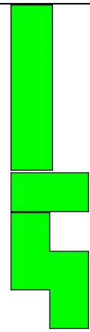
h’(n)= e(m)

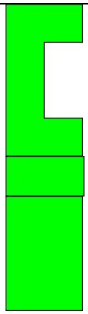
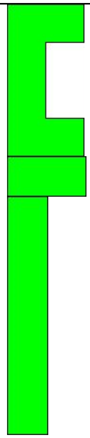
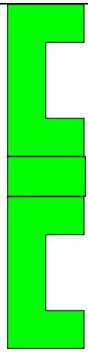
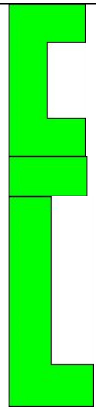
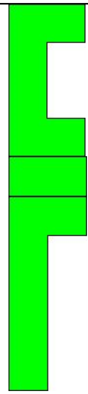
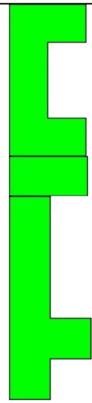
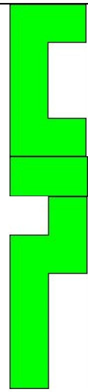
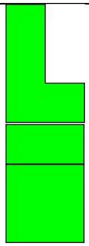
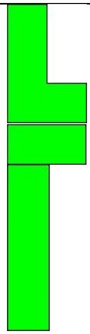
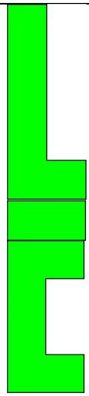
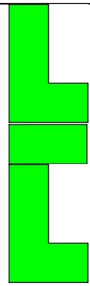
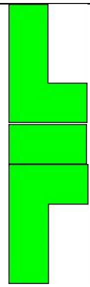
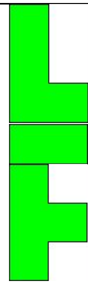
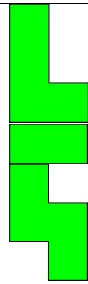
(To these values must be added the corresponding case (ii) values).

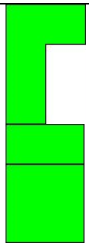
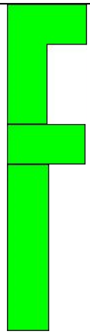
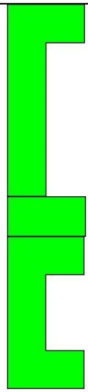
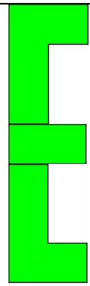
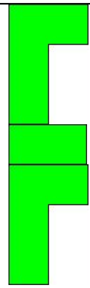
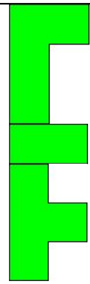
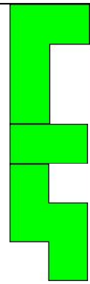
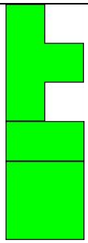
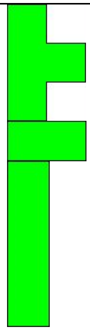
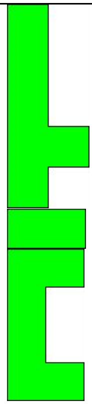
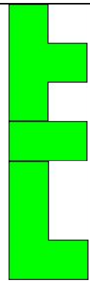
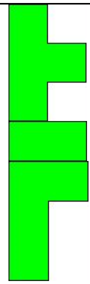
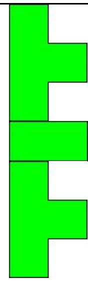
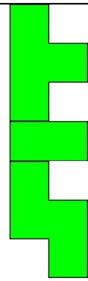
1. n is even, equal to 2m, for odd m, sub-case ii

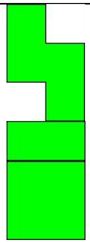
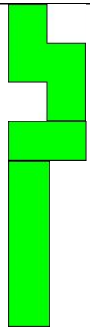
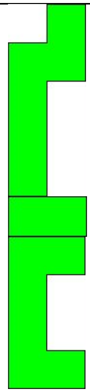
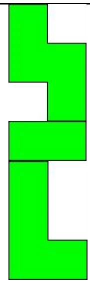
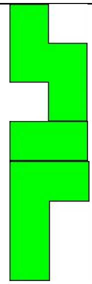
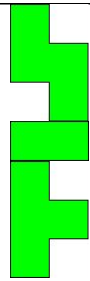
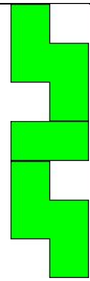
We now perform the same operation for sub-case (ii).

Above and below the line there will be a polyomino of size k=m-1, even.

	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a		N.a.		f:= 1		d:= D(k)		d:= e(k)		f:= f(k)		f:= G(k)		f:= H(k)
b		e:= 1		g’:= 1 h’:= 1		e:= 2D(k)		e:= 2e(k)		g:= f(k) h:= f(k)		g:= G(k) h:= G(k)		g:= H(k) h:= H(k)

d/d’		d:= D(k)		f:= 2D(k)		d’:= 2D(k) d:= 2D(k)*(D(k)-1)		d:= 2D(k)*e(k)		f:= 2D(k)*f(k)		f:= 2D(k)*G(k)		f:= 2D(k)*H(k)
e		e:= e(k)		g:= e(k) h:= e(k)		e:= 2e(k)*D(k)		e:= 2e(k)*e(k)		g’:= e(k) h’:= e(k) g:= e(k)*(e(k)-1) h:= e(k)*(e(k)-1)		g:= e(k)*G(k) h:= e(k)*G(k)		g:= e(k)*H(k) h:= e(k)*H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

f		d:= f(k)		f:= 2f(k)		d:= 2f(k)*D(k)		d':= 2f(k) d:= 2f(k)*(f(k) -1)		f:= 2f(k)*f(k)		f:= 2f(k)* G(k)		f:= 2f(k)*H(k)
g/g'		e:= G(k)		g:= G(k) h:= G(k)		e:= 2G(k)*D (k)		e:= 2G(k)*e(k)		g:= G(k)*f(k) h:= G(k)*f(k)		g':= G(k) h':= G(k) g:= G(k)*(G(k)-1) h:= G(k)*(G(k)-1)		g:= G(k)*H(k) h:= G(k)*H(k)
	a		b		d/d'		e		f		g/g'		h/h'	

h/h’		e:= H(k)		g:= H(k) h:= H(k)		e:= 2H(k)*D(k)		e:= 2H(k)*e(k)		g:= H(k)*f(k) h:= H(k)*f(k)		g:= H(k)*G(k) h:= H(k)*G(k)		h’:= H(k) g’:= H(k) g:= H(k)*(H(k)-1) h:= H(k)*(H(k)-1)
	a		b		d/d’		e		f		g/g’		h/h’	

n >= 4 (which means, for odd m, n >= 6; but we are using variation on same functions for even m)

d(n)= 2*D(k)*(D(k)-1)+2*D(k)*e(k)+2*f(k)*(f(k)-1)+2f(k)*D(k)+D(k)+D(k)+e(k)+f(k)

d'(n)= 2*D(k) + 2*f(k)

e(n)= 1+2*D(k)+2*e(k)+2*e(k)*D(k)+2*e(k)*e(k)+2*G(k)*D(k)

+2*G(k)*e(k)+2*H(k)*D(k)+2*H(k)*e(k)+e(k)+G(k)+H(k)

f(n)= 1 +2*D(k)+2*D(k)*f(k)+2*D(k)*G(k)+2*D(k)*H(k) +2*f(k)+2*f(k)*f(k)

+2*f(k)*G(k)+2*f(k)*H(k) +f(k)+G(k)+H(k)

g(n)= e(k) +e(k)*(e(k)-1) +e(k)*G(k) +e(k)*H(k) +f(k) +G(k) +G(k) +G(k)*(G(k)-1) +G(k)*f(k)

+G(k)*H(k) +H(k) +H(k) +H(k)*(H(k)-1) +H(k)*f(k) +H(k)*G(k)

g'(n)= 1 + e(k) + G(k) + H(k)

h(n)= e(k)+e(k)*(e(k)-1)+e(k)*G(k)+e(k)*H(k) +f(k)+G(k)+G(k)+G(k)*(G(k)-1)+G(k)*f(k)+

G(k)*H(k) +H(k) +H(k)+H(k)*(H(k)-1) +H(k)*f(k)+H(k)*G(k)

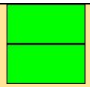
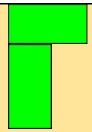
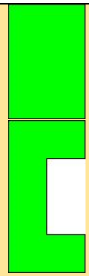
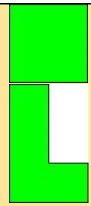
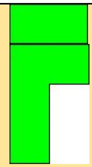
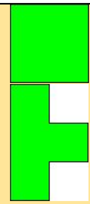
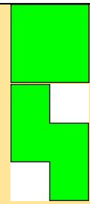
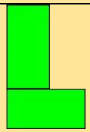
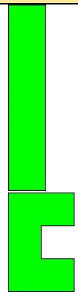
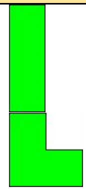
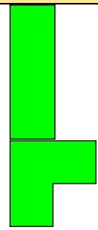
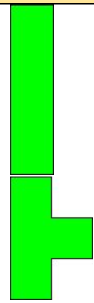
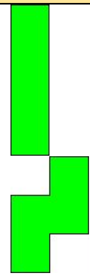
h'(n)= 1+ e(k) + G(k) + H(k)

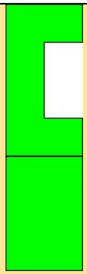
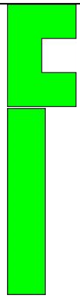
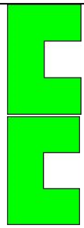
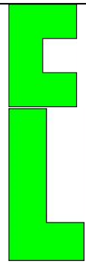
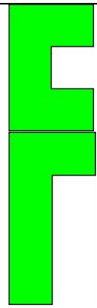
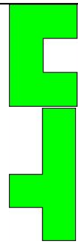
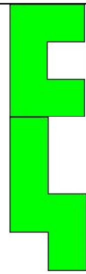
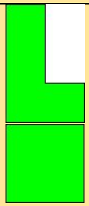
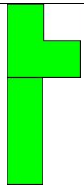
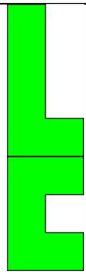
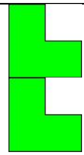
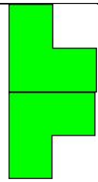
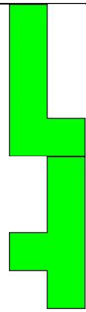
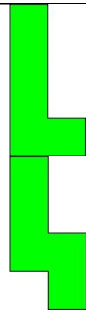
2. n is even, equal to 2m, for even m

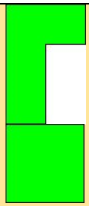
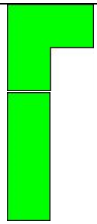
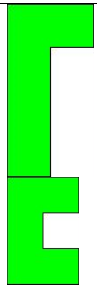
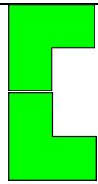
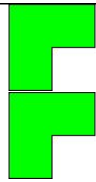
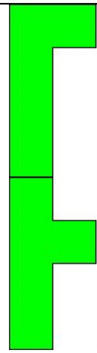
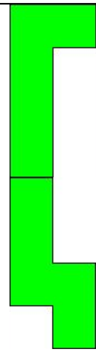
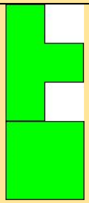
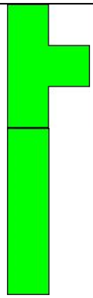
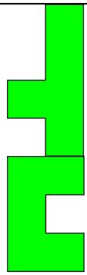
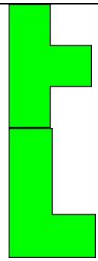
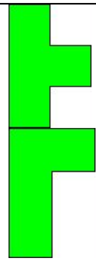
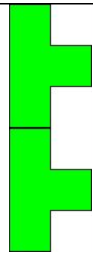
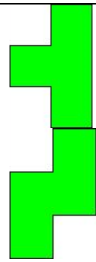
We will repeat the 1i and 1ii tables, highlighting the differences.

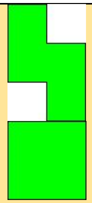
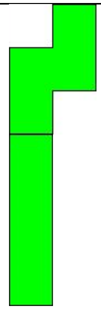
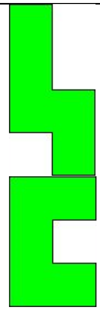
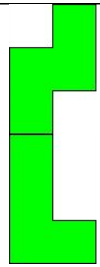
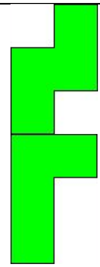
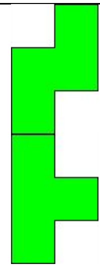
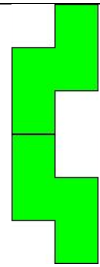
In the 1i table, the “a” column and row are added, and so the terms must be added to the formulae.

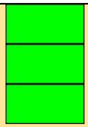
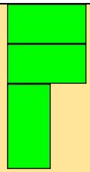
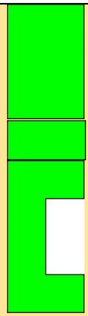
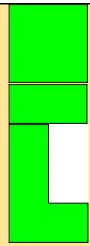
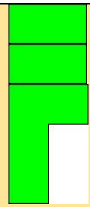
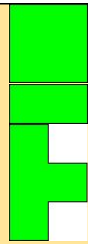
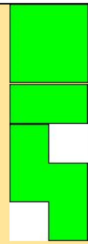
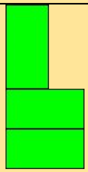
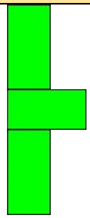
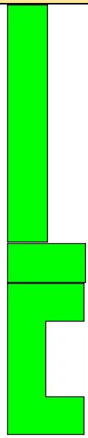
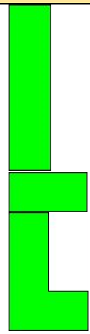
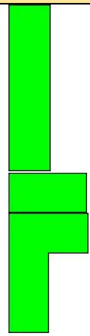
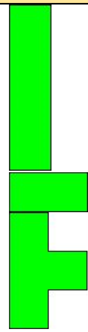
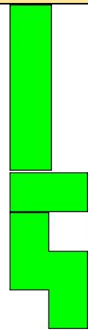
In the 1ii table, the “a” column and row must be removed, and so the terms must be subtracted from the formulae.

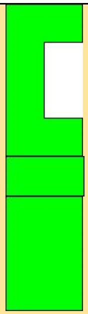
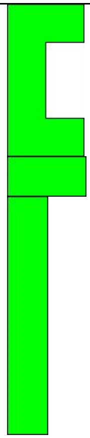
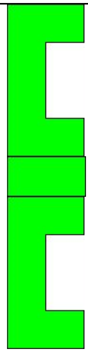
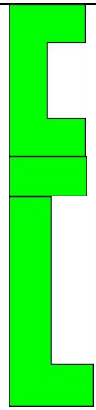
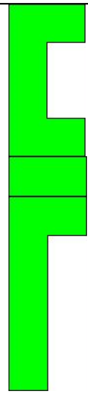
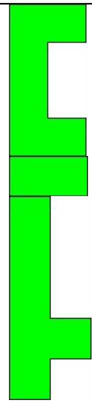
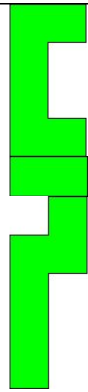
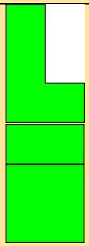
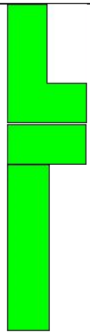
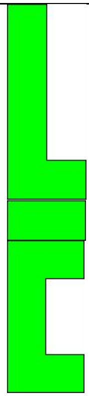
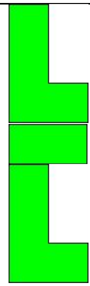
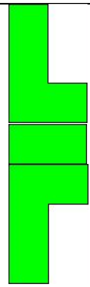
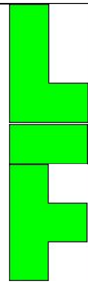
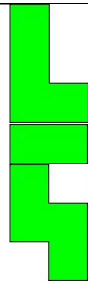
	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a		N.a.		f: 1		d: D(m)		d: e(m)		f: f(m)		f: G(m)		f: H(m)
b		e: 1	N.a.			e: 2D(m)		e: e(m)		g: f(m) h: f(m)		g: G(m)		h: H(m)

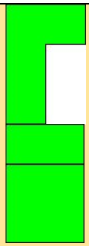
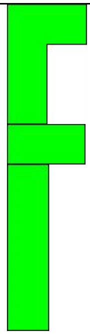
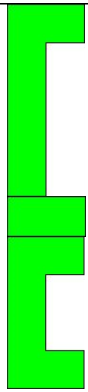
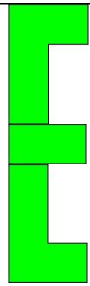
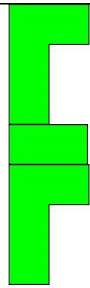
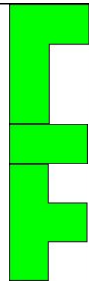
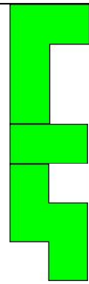
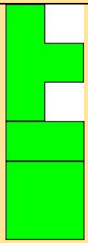
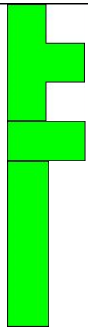
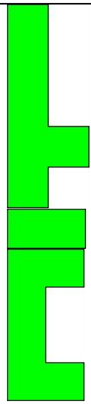
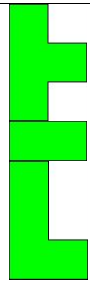
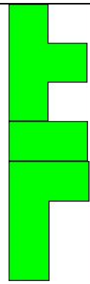
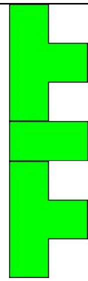
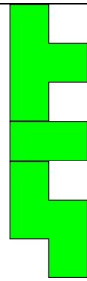
d/d’		d: D(m)		f: 2D(m)		d’: 2D(m) d: 2D(m)(D(m)-1)		d: 2D(m)e(m)		f: 2D(m)f(m)		f: 2D(m) G(m)		f: 2D(m)H(m)
e		e: e(m)		g: e(m) h: e(m)		e: 2e(m)D(m)		e: 2e(m)e(m)		g’: e(m) h’: e(m) g: e(m)(e(m) - 1) h: e(m)(e(m)- 1)		g: e(m)G(m) h: e(m)G(m)		g: e(m)H(m) h: e(m)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

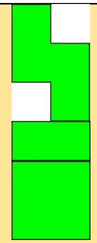
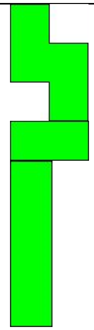
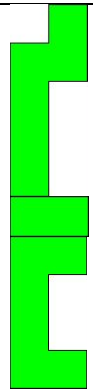
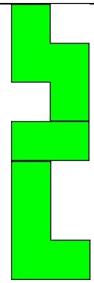
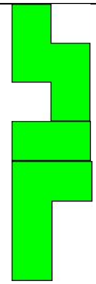
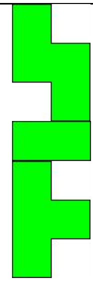
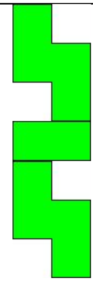
f		d: f(m)		f: f(m)		d: 2f(m)D(m)		d’: e(m) d: e(m)(f(m)-1)		f: 2f(m)f(m)		f: f(m)G(m)		f: f(m)H(m)
g/g’		e: G(m)		g: G(m)		e: 2G(m)D(m)		e: G(m)e(m)		g: G(m)f(m) h: G(m)f(m)		g: G(m)(G(m)-1) g’: G(m)		h: G(m)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

h/h’		e: H(m)		h: H(m)		e: 2H(m)D(m)		e: H(m)e(m)		g: H(m)f(m) h: H(m)f(m)		h: H(m)G(m)		g’: H(m) g: H(m)(H(m)-1)
	a		b		d/d’		e		f		g/g’		h/h’	

	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a		N.a.		~~f: 1~~		~~d: D(k)~~		~~d: e(k)~~		~~f: f(k)~~		~~f: G(k)~~		~~f: H(k)~~
b		~~e: 1~~		g’: 1 h’: 1		e: 2D(k)		e: 2e(k)		g: f(k) h: f(k)		g: G(k) h: G(k)		g: H(k) h: H(k)

d/d'		~~d: D(k)~~		f: 2D(k)		d': 2D(k) d: 2D(k)(D(k)-1)		d: 2D(k)e(k)		f: 2D(k)f(k)		f: 2D(k)G(k)		f: 2D(k)H(k)
e		~~e: e(k)~~		g: e(k) h: e(k)		e: 2e(k)D(k)		e: 2e(k)e(k)		g': e(k) h': e(k) g: e(k)(e(k)-1) h: e(k)(e(k)-1)		g: e(k)G(k) h: e(k)G(k)		g: e(k)H(k) h: e(k)H(k)
	a		b		d/d'		e		f		g/g'		h/h'	

f		~~d: f(k)~~		f: 2f(k)		d: 2f(k)D(k)		d': 2f(k) d: 2f(k)(f(k)-1)		f: 2f(k)f(k)		f: 2f(k)G(k)		f: 2f(k)H(k)
g/g'		~~e: G(k)~~		g: G(k) h: G(k)		e: 2G(k)D(k)		e: 2G(k)e(k)		g: G(k)f(k) h: G(k)f(k)		g': G(k) h': G(k) g: G(k)(G(k)-1) h: G(k)(G(k)-1)		g: G(k)H(k) h: G(k)H(k)
	a		b		d/d'		e		f		g/g'		h/h'	

h/h’		~~e: H(k)~~		g: H(k) h: H(k)		e: 2H(k)D(k)		e: 2H(k)e(k)		g: H(k)f(k) h: H(k)f(k)		g: H(k)G(k) h: H(k)G(k)		h’: H(k) g’: H(k) g: H(k)(H(k)-1) h: H(k)(H(k)-1)
	a		b		d/d’		e		f		g/g’		h/h’	

So we need to add in:

d: D(m)
d: D(m)
d: e(m)
d: f(m)
e: 1
e: e(m)
e: G(m)
e: H(m)
f: 1
f: f(m)
f: G(m)
f: H(m)

And remove:

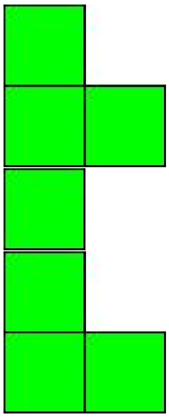
d: D(k)
d: D(k)
d: e(k)
d: f(k)
e: 1
e: e(k)
e: G(k)
e: H(k)
f: 1
f: f(k)
f: G(k)
f: H(k)

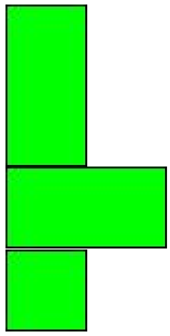
3. n is odd = 2m + 1

Two cases.

i. Middle square is single. Above and below the single square there are polyominoes of size m, which is odd or even.

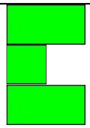
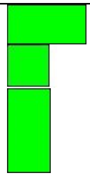
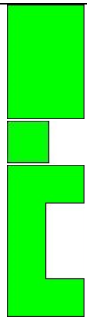
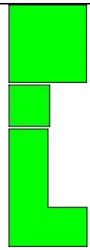
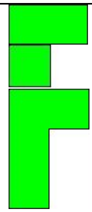
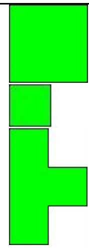
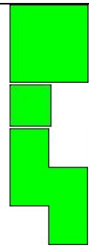
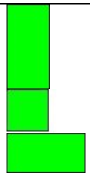
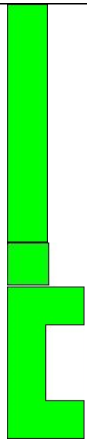
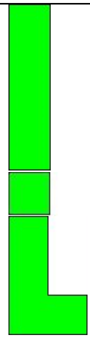
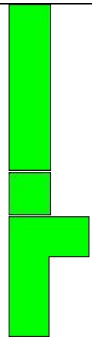
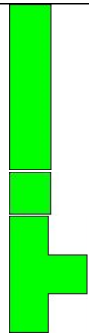
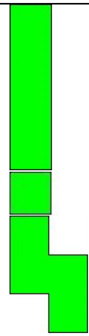
ii. In the “middle” there is a horizontal domino. Above is a polyomino of size m-1, and below one of size m, or vice versa.

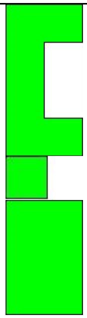
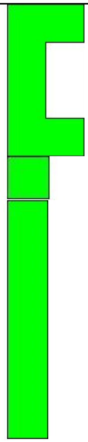
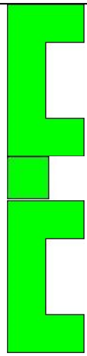
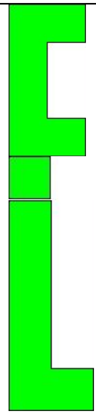
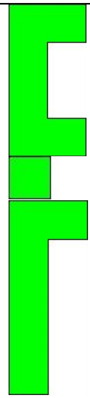
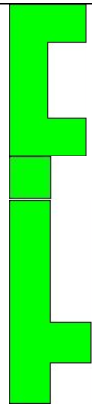
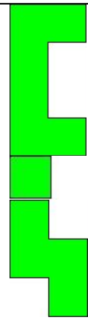
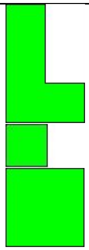
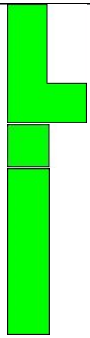
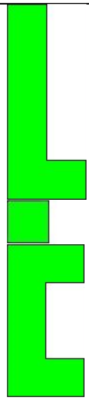
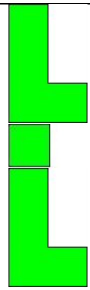
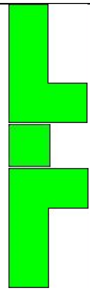
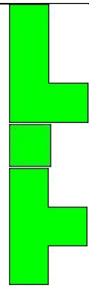
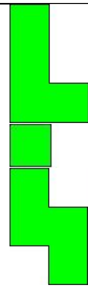


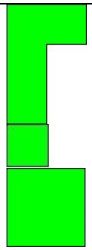
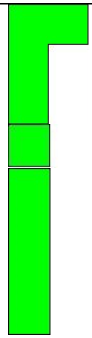
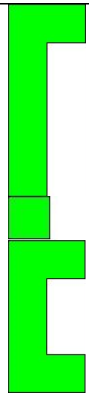
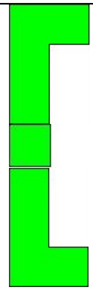
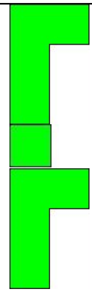
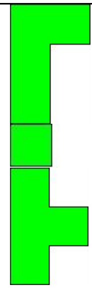
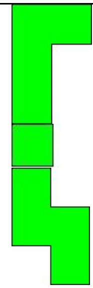
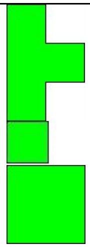
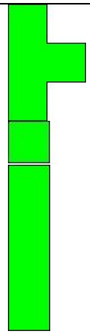
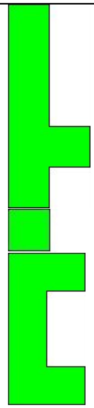
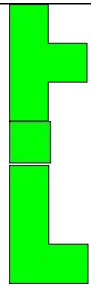
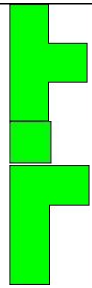
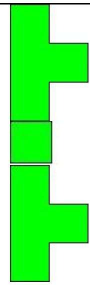
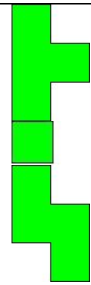


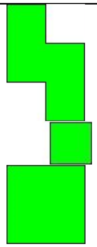
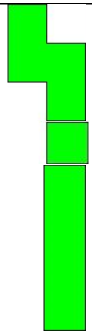
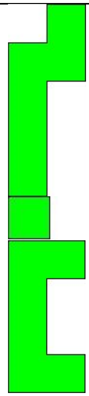
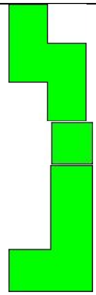
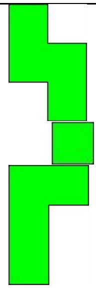
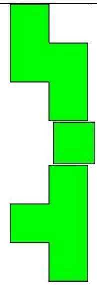
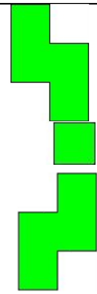
3. n is odd = 2m + 1, case i

First case i, which could have m even or odd. See specific notes in the table.

	BELOW													
ABOVE	a (only m even)		b		d/d’		e		f		g/g’		h/h’	
a (only m even)		d’:=1		f:= 1		d:= 2*D(m)		d:= e(m)		f:=2* f(m)		f:= G(m)		f:= H(m)
b		e:= 1	N.a.			e:= 2D(m)		e:= e(m)		g:= f(m) h:= f(m)		g:= G(m)		h:= H(m)

d/d’		d:= 2*D(m)		f:= 2*D(m)		d’:= 2*D(m) d:= 4*D(m)*(D(m)-1) + 2*D(m)		d:= 2*D(m)*e(m)		f:= 4*D(m)*f(m)		f:= 2*D(m)*G(m)		f:= 2*D(m)*H(m)
e		e:= 2*e(m)		g:= e(m) h:= e(m)		e:= 4*e(m)*D(m)		e:= 2*e(m)*e(m)		g’:= 2*e(m) g:= 2*e(m)*(e(m)-1) h:= 2*e(m)*e(m)		g:= e(m)*G(m) h:= e(m)*G(m)		g:= e(m)*H(m) h:= e(m)*H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

f		d:= f(m)		f:= f(m)		d:= 2f(m)*D (m)		d':= f(m) d:= f(m)*(f(m)-1)		f:= 2f(m)*f(m)		f:= f(m)*G (m)		f:= f(m)*H(m)
g/g'		e:= G(m)		g:= G(m)		e:= 2G(m)* D(m)		e:= G(m)*e(m)		g:= G(m)*f(m) h:= G(m)*f(m)		g':= G(m) g:= G(m)*(G(m)- 1)		h:= G(m)*H(m)
	a		b		d/d'		e		f		g/g'		h/h'	

h/h’		e:= H(m)		h:= H(m)		e:= 2H(m)* D(m)		e:= H(m)*e(m)		g:= H(m)*f(m) h:= H(m)*f(m)		h:= H(m)* G(m)		g’:= H(m) g:= H(m)*(H(m)-1)
	a		b		d/d’		e		f		g/g’		h/h’	

The following formulae represent the non-optional parts of the above table.

n odd, n=2m+1, m odd

d(n)= 2*D(m)*e(m) +2f(m)*D(m) +4*D(m)*(D(m)-1) + 2*D(m) +f(m)*(f(m)-1)

d'(n)= 2*D(m) + f(m)

e(n)= 2*e(m)*e(m) +2D(m) +2G(m)*D(m) +2H(m)*D(m) +4*e(m)*D(m) +e(m) +G(m)*e(m) +H(m)*e(m)

f(n)= 2*D(m) +2*D(m)*G(m) +2*D(m)*H(m) +2f(m)*f(m) +4*D(m)*f(m) +f(m) +f(m)*G(m) +f(m)*H(m)

g(n)= 2*e(m)*(e(m)-1) +e(m) +e(m)*G(m) +e(m)*H(m) +f(m) +G(m) +G(m) +G(m)*(G(m)-1) +G(m)*f(m) +H(m)*(H(m)-1) +H(m)*f(m)

g'(n)= 2*e(m) + G(m) + H(m)

h(n)= 2*e(m)*e(m) +e(m) +e(m)*G(m) +e(m)*H(m) +f(m) +G(m)*f(m) +G(m)*H(m) +H(m) +H(m) +H(m)*G(m) +H(m)*f(m)

If m is even, we must add in:

d(n)= 2*D(m) +e(m) +2*D(m) +f(m)

d'(n)=1

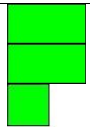
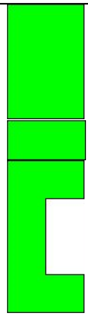
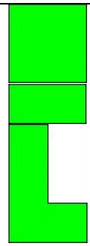
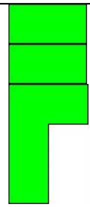
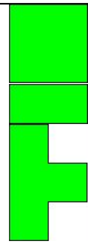
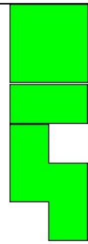
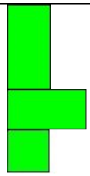
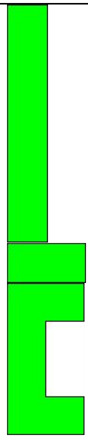
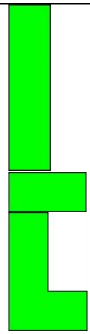
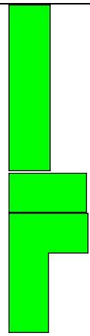
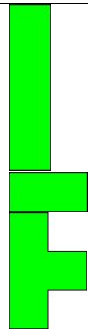
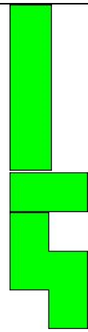
e(n)= 2*e(m) +G(m) +H(m) +1

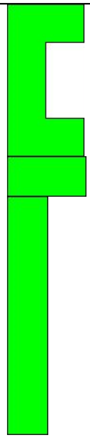
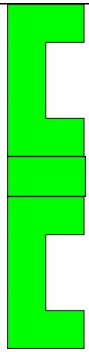
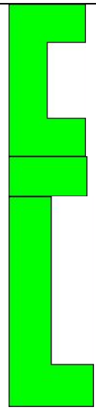
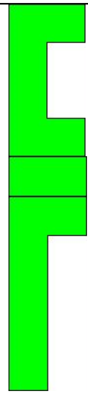
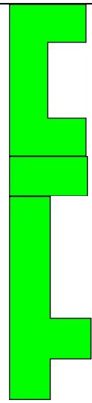
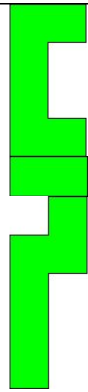
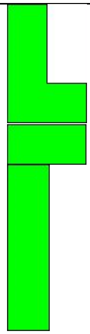
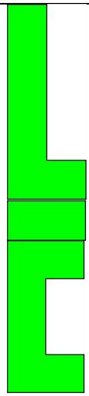
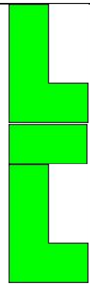
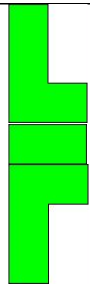
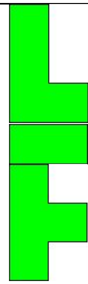
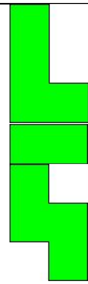
f(n)=2* f(m) +G(m) +H(m) +1

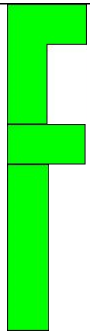
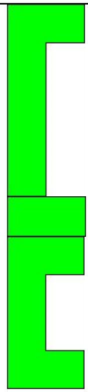
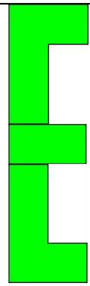
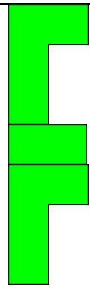
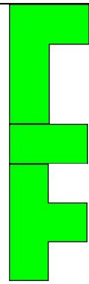
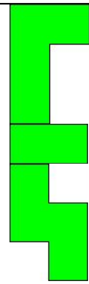
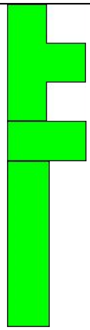
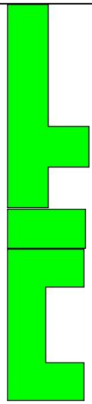
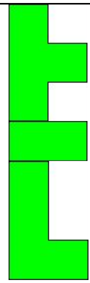
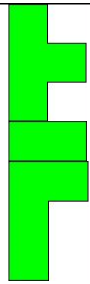
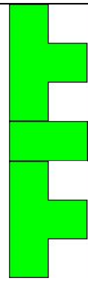
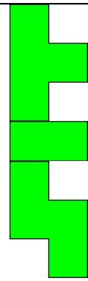
3. n is odd = 2m + 1, case ii

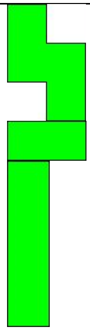
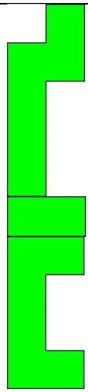
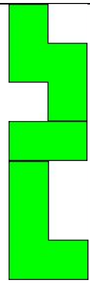
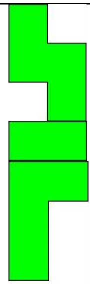
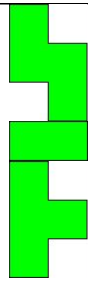
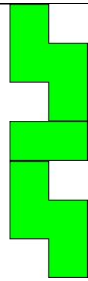
k = m-1

Table 1: m above, k below, m even, k odd.

	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a	N.a.			f: 1		d: D(k)		d: e(k)		f: f(k)		f: G(k)		f: H(k)
b	N.a.			g: 1 h: 1		e: 2D(k)		e: 2e(k)		g: f(k) h: f(k)		g: G(k) h: G(k)		g: H(k) h: H(k)

d/d’				f: 2D(m)		d: 2D(m)D(k)		d: 2D(m)e(k)		f: 2D(m)f(k)		f: 2D(m) G(k)		f: 2D(m)H(k)
e				g: e(m) h: e(m)		e: 2e(m)D(k)		e: 2e(m)e(k)		g: e(m)e(k) h: e(m)e(k)		g: e(m)G(k) h: e(m)G(k)		g: e(m)H(k) h: e(m)H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

f				f: 2f(m)		d: 2f(m)D(k)		d: 2f(m)f(k)		f: 2f(m)f(k)		f: 2f(m)G(k)		f: 2f(m)H(k)
g/g’				g: G(m) h: G(m)		e: 2G(m)D(k)		e: 2G(m)e(k)		g: G(m)f(k) h: G(m)f(k)		g: G(m)G(k) h: G(m)G(k)		g: G(m)H(k) h: G(m)H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

h/h’				g: H(m) h: H(m)		e: 2H(m)D(k)		e: 2H(m)e(k)		g: H(m)f(k) h: H(m)f(k)		g: H(m)G(k) h: H(m)G(k)		g: H(m)H(k) h: H(m)H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

The resulting formulae are

(odd, even, m above)

d(n)= 2D(m)D(k) + 2D(m)e(k) + 2f(m)D(k) + 2f(m)f(k) + D(k) + e(k)

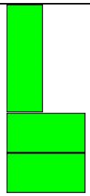
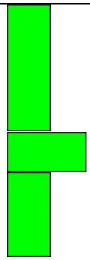
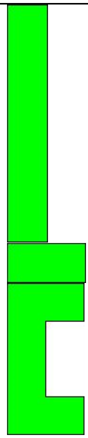
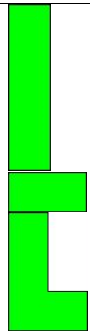
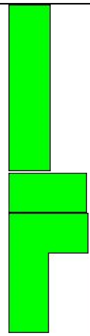
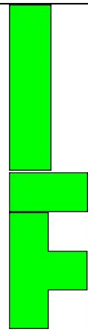
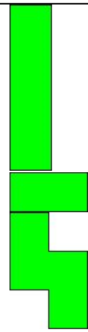
e(n)= 2D(k) + 2e(k) + 2e(m)D(k) + 2e(m)e(k) + 2G(m)D(k) + 2G(m)e(k) + 2H(m)D(k) + 2H(m)e(k)

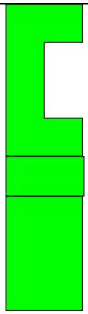
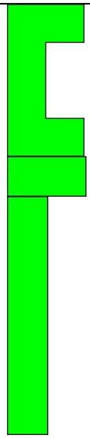
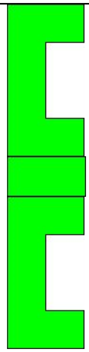
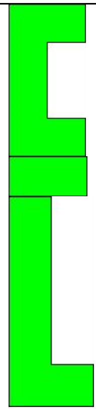
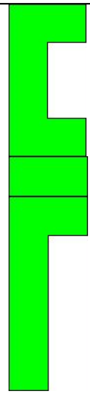
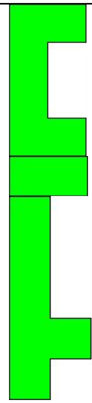
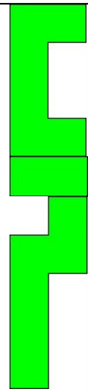
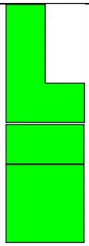
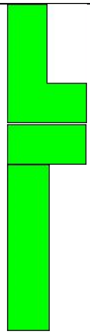
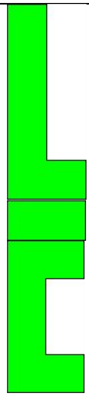
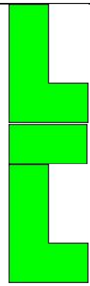
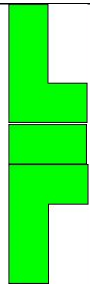
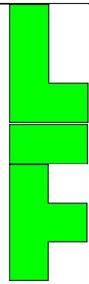
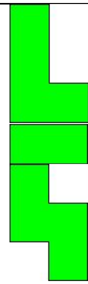
f(n)= 1 + 2D(m) + 2D(m)f(k) + 2D(m)G(k) + 2D(m)H(k) + 2f(m) + 2f(m)f(k) + 2f(m)G(k) + 2f(m)H(k) + f(k) + G(k) + H(k)

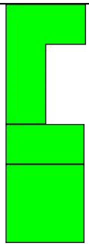
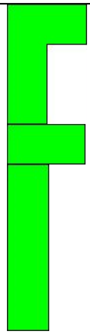
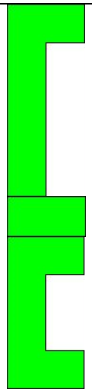
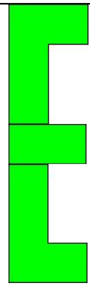
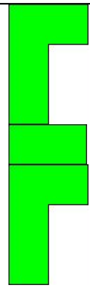
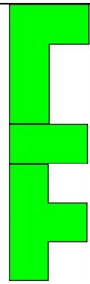
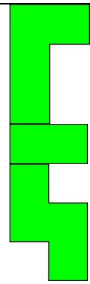
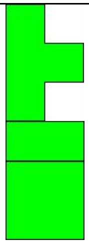
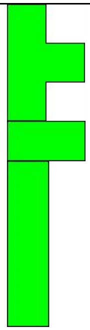
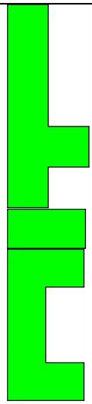
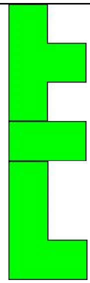
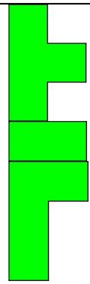
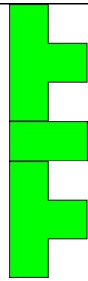
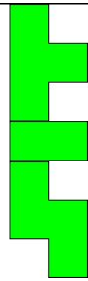
g(n)= 1 + e(m) + e(m)e(k) + e(m)G(k) + e(m)H(k) + f(k) + G(k) + G(m) + G(m)f(k) + G(m)G(k) + G(m)H(k) + H(k) + H(m) + H(m)f(k) + H(m)G(k) + H(m)H(k)

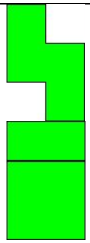
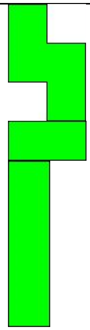
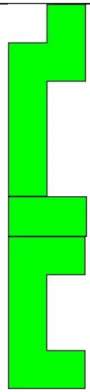
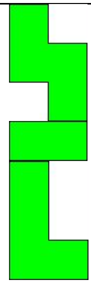
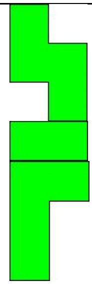
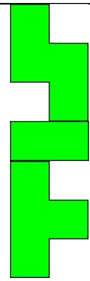
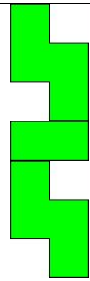
h(n)= 1 + e(m) + e(m)e(k) + e(m)G(k) + e(m)H(k) + f(k) + G(k) + G(m) + G(m)f(k) + G(m)G(k) + G(m)H(k) + H(k) + H(m) + H(m)G(k) + H(m)H(k) + H(m)f(k)

Table 2: m above, k below, m odd, k even

	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a														
b		e: 1		g: 1 h: 1		e: 2D(k)		e: 2e(k)		g: f(k) h: f(k)		g: G(k) h: G(k)		g: H(k) h: H(k)

d/d’		d: D(m)		f: 2D(m)		d: 2D(m)D(k)		d: 2D(m)e(k)		f: 2D(m)f(k)		f: 2D(m) G(k)		f: 2D(m)H(k)
e		e: e(m)		g: e(m) h: e(m)		e: 2e(m)D(k)		e: 2e(m)e(k)		g: e(m)e(k) h: e(m)e(k)		g: e(m)G(k) h: e(m)G(k)		g: e(m)H(k) h: e(m)H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

f		d: f(m)		f: 2f(m)		d: 2f(m)D(k)		d: 2f(m)f(k)		f: 2f(m)f(k)		f: 2f(m)G(k)		f: 2f(m)H(k)
g/g’		e: G(m)		g: G(m) h: G(m)		e: 2G(m)D(k)		e: 2G(m)e(k)		g: G(m)f(k) h: G(m)f(k)		g: G(m)G(k) h: G(m)G(k)		g: G(m)H(k) h: G(m)H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

h/h’		e: H(m)		g: H(m) h: H(m)		e: 2H(m)D(k)		e: 2H(m)e(k)		g: H(m)f(k) h: H(m)f(k)		g: H(m)G(k) h: H(m)G(k)		g: H(m)H(k) h: H(m)H(k)
	a		b		d/d’		e		f		g/g’		h/h’	

$$d(n)= 2D(m)D(k) +2D(m)e(k) +2f(m)D(k) +2f(m)f(k) +D(m) +f(m)$$

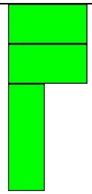
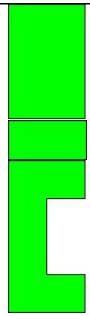
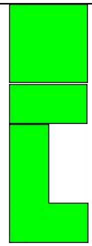
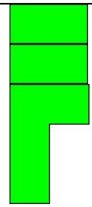
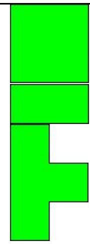
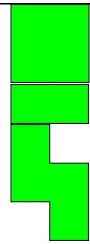
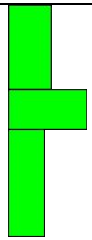
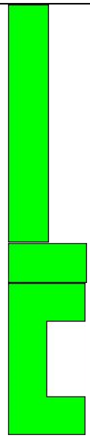
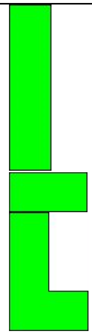
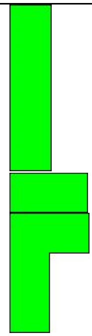
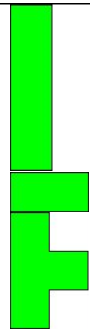
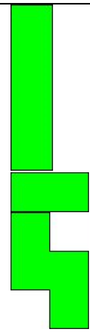
$$e(n)= 1 +2D(k) +2e(k) +2e(m)D(k) +2e(m)e(k) +2G(m)D(k) +2G(m)e(k) +2H(m)D(k) +2H(m)e(k) +e(m) +G(m) +H(m)$$

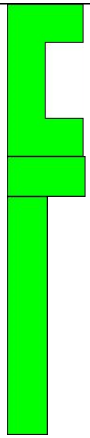
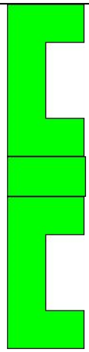
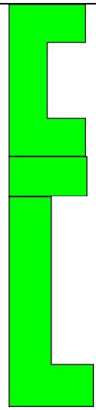
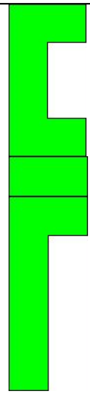
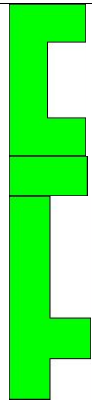
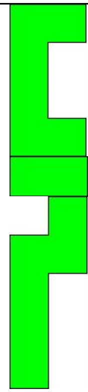
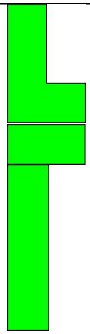
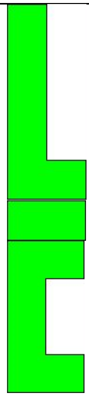
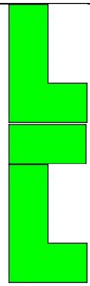
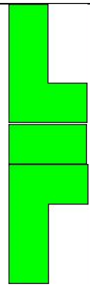
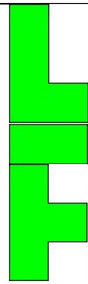
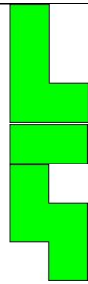
$$f(n)= 2D(m) +2D(m)f(k) +2D(m)G(k) +2D(m)H(k) +2f(m) +2f(m)f(k) +2f(m)G(k) +2f(m)H(k)$$

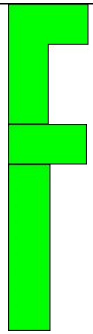
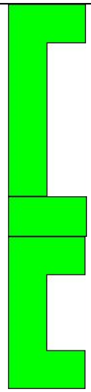
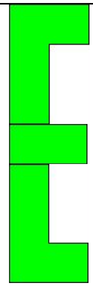
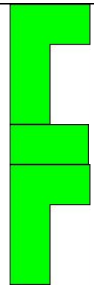
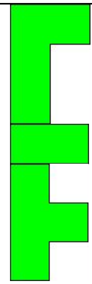
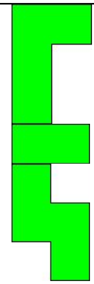
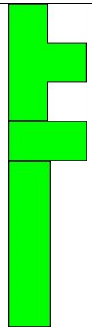
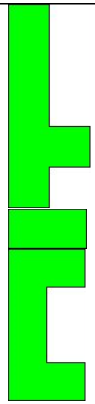
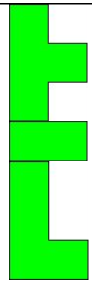
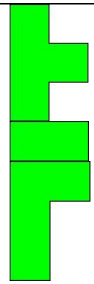
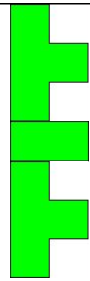
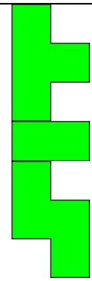
$$g(n)= e(m) +e(m)e(k) +e(m)G(k) +e(m)H(k) +f(k) +G(k) +G(m) +G(m)f(k) +G(m)G(k) +G(m)H(k) +H(k) +H(m) +H(m)f(k) +H(m)G(k) +H(m)H(k) + 1$$

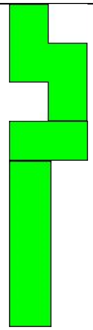
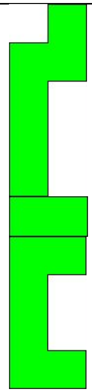
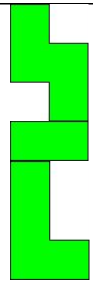
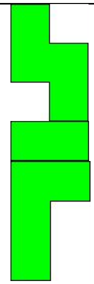
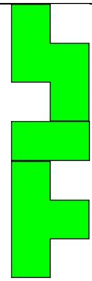
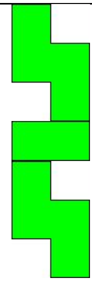
$$h(n)= e(m) +e(m)e(k) +e(m)G(k) +e(m)H(k) +f(k) +G(k) +G(m) +G(m)f(k) +G(m)G(k) +G(m)H(k) +H(k) +H(m) +H(m)G(k) +H(m)H(k) +H(m)f(k) + 1$$

Table 3: k above, m below, k even, m odd.

	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a	N.a.			f: 1		d: D(m)		d: e(m)		f: f(m)		f: G(m)		f: H(m)
b	N.a.			g: 1 (k > 0) h: 1 (k > 0)		e: 2D(m)		e: 2e(m)		g: f(m) h: f(m)		g: G(m) h: G(m)		g: H(m) h: H(m)

d/d’				f: 2D(k)		d: 2D(k)D(m)		d: 2D(k)e(m)		f: 2D(k)f(m)		f: 2D(k)G(m)		f: 2D(k)H(m)
e				g: e(k) h: e(k)		e: 2e(k)D(m)		e: 2e(k)e(m)		g: e(k)e(m) h: e(k)e(m)		g: e(k)G(m) h: e(k)G(m)		g: e(k)H(m) h: e(k)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

f				f: 2f(k)		d: 2f(k)D(m)		d: 2f(k)f(m)		f: 2f(k)f(m)		f: 2f(k)G(m)		f: 2f(k)H(m)
g/g’				g: G(k) h: G(k)		e: 2G(k)D(m)		e: 2G(k)e(m)		g: G(k)f(m) h: G(k)f(m)		g: G(k)G(m) h: G(k)G(m)		g: G(k)H(m) h: G(k)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

h/h’				g: H(k) h: H(k)		e: 2H(k)D(m)		e: 2H(k)e(m)		g: H(k)f(m) h: H(k)f(m)		g: H(k)G(m) h: H(k)G(m)		g: H(k)H(m) h: H(k)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

$d(n) = 2D(k)D(m) + 2D(k)e(m) + 2f(k)D(m) + 2f(k)f(m) + D(m) + e(m)$

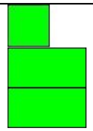
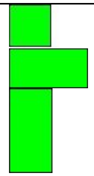
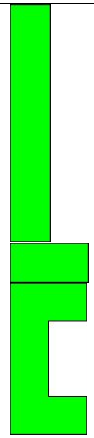
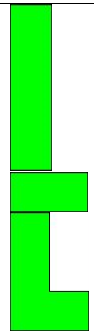
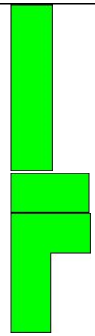
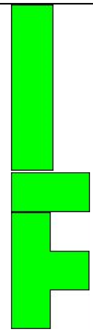
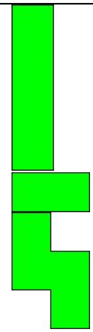
$e(n) = 2D(m) + 2e(k)D(m) + 2e(k)e(m) + 2e(m) + 2G(k)D(m) + 2G(k)e(m) + 2H(k)D(m) + 2H(k)e(m)$

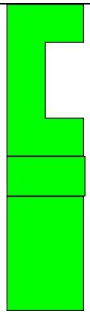
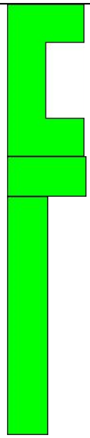
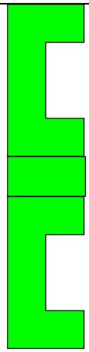
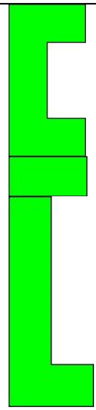
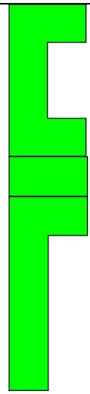
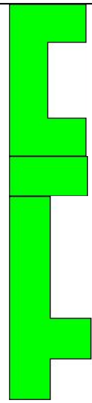
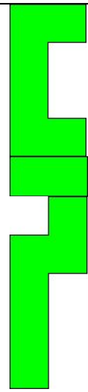
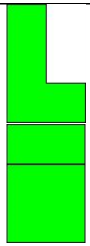
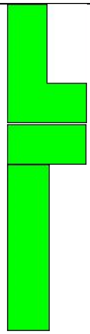
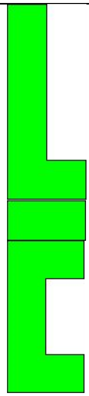
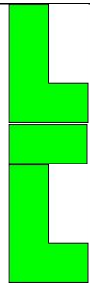
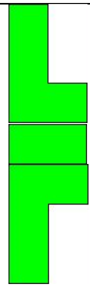
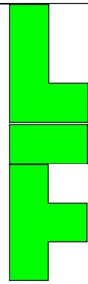
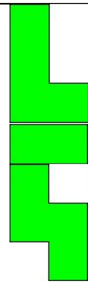
$f(n) = 1 \text{ (if } k > 0) + 2D(k) + 2D(k)f(m) + 2D(k)G(m) + 2D(k)H(m) + 2f(k) + 2f(k)f(m) + 2f(k)G(m) + 2f(k)H(m) + f(m) + G(m) + H(m)$

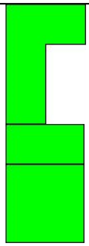
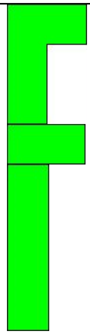
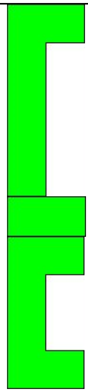
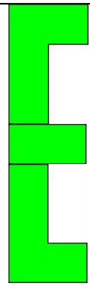
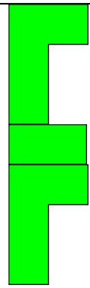
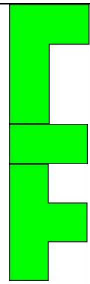
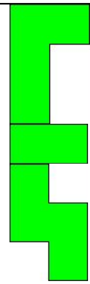
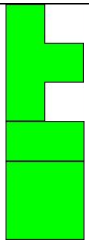
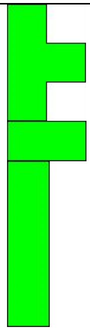
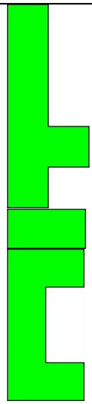
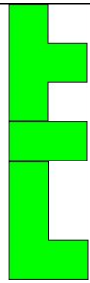
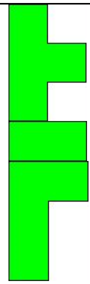
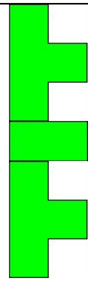
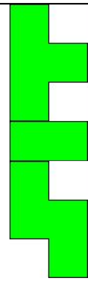
$g(n) = 1 \text{ (if } k > 0) + e(k) + e(k)e(m) + e(k)G(m) + e(k)H(m) + f(m) + G(k) + G(k)f(m) + G(k)G(m) + G(k)H(m) + G(m) + H(k) + H(k)f(m) + H(k)G(m) + H(k)H(m) + H(m)$

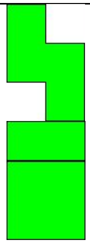
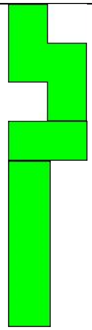
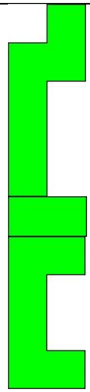
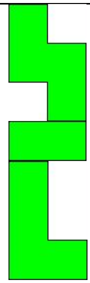
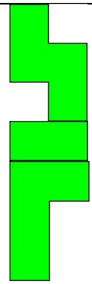
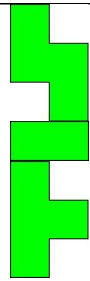
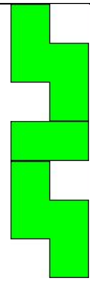
$h(n) = 1 + e(k) + e(k)e(m) + e(k)G(m) + e(k)H(m) + f(m) + G(k) + G(k)f(m) + G(k)G(m) + G(k)H(m) + G(m) + H(k) + H(k)G(m) + H(k)H(m) + H(m) + H(k)f(m)$

Table 4: k above, m below, k odd, m even

	BELOW													
ABOVE	a		b		d/d’		e		f		g/g’		h/h’	
a														
b		e: 1		g: 1 h: 1		e: 2D(m)		e: 2e(m)		g: f(m) h: f(m)		g: G(m) h: G(m)		g: H(m) h: H(m)

d/d’		d: D(k)		f: 2D(k)		d: 2D(k)D(m)		d: 2D(k)e(m)		f: 2D(k)f(m)		f: 2D(k)G(m)		f: 2D(k)H(m)
e		e: e(k)		g: e(k) h: e(k)		e: 2e(k)D(m)		e: 2e(k)e(m)		g: e(k)e(m) h: e(k)e(m)		g: e(k)G(m) h: e(k)G(m)		g: e(k)H(m) h: e(k)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

f		d: f(k)		f: 2f(k)		d: 2f(k)D(m)		d: 2f(k)f(m)		f: 2f(k)f(m)		f: 2f(k)G(m)		f: 2f(k)H(m)
g/g’		e: G(k)		g: G(k) h: G(k)		e: 2G(k)D(m)		e: 2G(k)e(m)		g: G(k)f(m) h: G(k)f(m)		g: G(k)G(m) h: G(k)G(m)		g: G(k)H(m) h: G(k)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

h/h’		e: H(k)		g: H(k) h: H(k)		e: 2H(k)D(m)		e: 2H(k)e(m)		g: H(k)f(m) h: H(k)f(m)		g: H(k)G(m) h: H(k)G(m)		g: H(k)H(m) h: H(k)H(m)
	a		b		d/d’		e		f		g/g’		h/h’	

$d(n) = 2D(k)D(m) + 2D(k)e(m) + 2f(k)D(m) + 2f(k)f(m) + D(k) + f(k)$

$e(n) = 1 + 2D(m) + 2e(k)D(m) + 2e(k)e(m) + 2e(m) + 2G(k)D(m) + 2G(k)e(m) + 2H(k)D(m) + 2H(k)e(m) + e(k) + G(k) + H(k)$

$f(n) = 2D(k) + 2D(k)f(m) + 2D(k)G(m) + 2D(k)H(m) + 2f(k) + 2f(k)f(m) + 2f(k)G(m) + 2f(k)H(m)$

$g(n) = e(k) + e(k)e(m) + e(k)G(m) + e(k)H(m) + f(m) + G(k) + G(k)f(m) + G(k)G(m) + G(k)H(m) + G(m) + H(k) + H(k)f(m) + H(k)G(m) + H(k)H(m) + H(m) + 1$

$h(n) = e(k) + e(k)e(m) + e(k)G(m) + e(k)H(m) + f(m) + G(k) + G(k)f(m) + G(k)G(m) + G(k)H(m) + G(m) + H(k) + H(k)G(m) + H(k)H(m) + H(m) + H(k)f(m) + 1$