Solution. The problem is from [1] and inspired also by recent computations by Mark Rickert. Part (b) was solved by Joseph DeVincentis and Dan Dima (both found that sInverse(4) has 111 elements, working entirely by hand). Stephen Meskin proved that sInverse(4) is finite. Mark Rickert has computed s(n) for n up to 8,000,000. The sequence begins as follows, for n from 1 to 50:

0,2,4,3,5,4,6,3,6,5,6,4,6,4,4,4,6,4,7,4,4,5,6,4,8,4,6,4,7,4,6,5,7,6,4,4,7,6,4,4,7,4,5,5,4,5,6,4,6,5

I have posted Rickert's data at <http://stanwagon.com/public/1340RickertData.txt> . These stabilizing lengths are remarkably small: 17 is the largest s(n) for n≤8,000,000. This table shows the tallies of the s-values. Note the single occurrence of ∞! The 111 is the correct total count for 4 (and the smaller values are correct too). The later entries will continue to grow as n rises:

s(n)	frequency
0	1
2	1
3	2
4	111
5	268820
6	1311460
7	3398966
8	2275437
9	614110
10	106868
11	18841
12	4234
13	895
14	187
15	56
16	9
17	1
∞	1

As found in [1], there appears to be only one value for which s(n) does not become constant, and that is 1969. That was not known in 1969, as it was discovered only in 1993. For n = 1969 the matrix looks like this below: each row is a column of the modular Ackermann matrix, and only the initial 15 entries of each column are shown. A minor miracle happens in that A(5,2), a gigantic number, turns out to be divisible by 1969. This explains the first 0 that occurs below. It migrates to the (7,0) position and then the miracle of period 3 occurs. And the column after that then becomes 1698,0,0,... which generates 0, 1698, 0, 1698 .... and these two alternate forever.

column 0: {1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}

column 1: {2,3,4,5,6,7,8,9,10,11,12,13,14,15,16}
column 2: {3,5,7,9,11,13,15,17,19,21,23,25,27,29,31}
column 3: {5,13,29,61,125,253,509,1021,76,155,313,629,1261,556,1115}
column 4: {13,556,1774,1720,1303,1392,392,194,1247,1087,152,1470,1424,1258,1440}
column 5: {556,1698,0,13,1258,1698,0,13,1258,1698,0,13,1258,1698,0}
column 6: {1698,0,556,1258,0,556,1258,0,556,1258,0,556,1258,0,556}
column 7: {0,1698,1258,0,1698,1258,0,1698,1258,0,1698,1258,0,1698,1258}
column 8: {1698,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
column 9: {0,1698,0,1698,0,1698,0,1698,0,1698,0,1698,0,1698,0}
column 10: {1698,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

So this shows that 1969 does not lead to a constant column, but rather to two alternating columns. This is the only known value that does not lead to a constant column

As for part (b), here is Dima's solution: Let s(n) be the column index of the first constant column. The initial columns of the Ackermann function are y+1, y+2, 2y+3, 2^(y+3)-3. Therefore

B(n,0,y) = mod(y+1, n), B(n,1,y) = mod(y+2, n),
B(n,2,y) = mod(2y + 3, n), and B(n,3,y) = mod(2^(y+3)-3, n)

Hence for n>13: B(n,3,0) = 5, B(n,3,1) = 13. And then B(n,4,0) = B(n,3,1) = 13 and B(n,4,1) = B(n,3,B(n,4,0)) = B(n,3,13).

B(n,3,13) = mod(2^16-3, n) = mod(65533, n)

It follows that for any n larger than 65333, B(n,4,1)=65333 and B(n,4,0)=13, so the column is not constant. Therefore if s(n) = 4, n is at most 65333, proving that sInverse(4) is finite. At this point computation will get the requested set, but it can be done by hand as follows.

Consider first sInverse(k) for k = 0, 1, 2, 3:

s(n) = 0 only when n=1, because B(n,0,y) = mod(y+1,n)
s(n) ≠ 1 for any n because the first two entries in column 1 are (0,1) for n = 2, (2,0) for n = 3, and (2,3) for any larger n.
s(n) = 2 only when n=2. The first two entries in column 2 as n rises from 2 are **(1,1)**, (0,2), (3,1), {3,0}, {3,5), {3,5),{3,5),...
s(n) = 3 only when n=4 or 8. The first two entries in column 2 as n rises from 2 are
(1,1), (2,1), **(1,1)**, (0,3), (5,1), (5,6), **(5,5)**, (5,4), (5,3), (5,2) (5,1) ,(5,0), (5,13), (5,13), (5,13), (5,13),...
But let's analyze this another way,.

If s(n) = 3 then column 3 is constant so we have B(n,3,0) = B(n,3,1) (and this is sufficient for constancy).

But A(n,3,0)=5 and A(n,3,1)=13; so mod(13, n) = mod(5, n), so 5 === 13 (mod n) and that happens only for n = 1, 2, 4 and 8. But for 1 and 2, stabilization occurs earlier. So the set is {4, 8}.

The preceding analysis works for s(n) = 4. We would need: B(n,4,0) = B(n,4,1). When n≥14 this is the same as 13 === 65533 (mod n). This is simply asking for the divisors of 65520 that are ≥14; there are 108 of them. For n≤13, and n ≠ 1, 2, 4, or 8, one can check that the values 3, 6, 12 are the ones that work. So the total count is 108+3=111 and the set SInverse[4] consists of the divisors of 65520 except for 1,2,4,8,5,7,9,10,13. QED

Source: Source. Jon Froemke and Jerrold Grossman, Amer Math Monthly, 180, February 1993, 180-183.