1340. Modular Ackermann

Let s(n) be the number of steps for the mod-n Ackermann function to stabilize.

(a) True or False: For every n, s(n) is finite.

(b) How many values of n are there for which s(n) = 4?

Definitions. The Ackermann function A(x,y) (x, y nonnegative integers) is the classic example of a very fast-growing function:
A(0,0) = 1, A(1,1) = 3; A(2,2) = 7, A(3,3) = 61, A(4,4) = 2^(2^(2^(2^16))) - 3, or about 10^10^10^20000. The definition follows and the table below shows the values (the UpArrow notation denotes repeated exponentiation: 2 UpArrow 4 is 2^2^2^2, or 65536).

A(0,y) = y+1
A(x,0) = A(x-1, 1)
A(x,y) = A(x-1, A(x, y-1))) if x≥1 and y≥1.

Out[∘]=

$y=5$	6	7	13	253	$2\uparrow 8$	really huge
$y=4$	5	6	11	125	$2\uparrow 7$	really huge
$y=3$	4	5	9	61	$2^{10^{20000}}$ about $2\uparrow 6$	really huge
$y=2$	3	4	7	29	10^{20000} about $2\uparrow 5$	really huge
$y=1$	2	3	5	13	65533, about $2\uparrow 4$	$2\uparrow 65536$
$y=0$	1	2	3	5	$13 = 2^4 - 3 = 2\uparrow 3 - 3$	65533, about $2\uparrow 4$
	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$

Consider the mod-n version B[n,x,y], where each step is reduced modulo n to the least nonnegative residue. That can be defined as follows.

B(n,0,y) = mod(y+1, n)
B(n,x,0) = B(n, x-1, 1)
B(n,x,y) = B(n, x-1, B(n,x, y-1))) if x≥1 and y≥1.
Let B*[n] be the n-by-∞ matrix where the i'th column is {B(i,j), 0 ≤ j ≤ n-1}. For example the first eight columns (i = 0, 1, 2,...,7) of B*[13] are below. The bottom row is B(13, i, 0); the top row is B(13, i, 12).

Out[ ]=

$j=12$	0	1	1	5	3	5	9	9
$j=11$	12	0	12	1	2	9	9	9
$j=10$	11	12	10	12	6	5	9	9
$j=9$	10	11	8	11	5	9	9	9
$j=8$	9	10	6	4	0	5	9	9
$j=7$	8	9	4	7	1	9	9	9
$j=6$	7	8	2	2	11	5	9	9
$j=5$	6	7	0	6	9	9	9	9
$j=4$	5	6	11	8	3	5	9	9
$j=3$	4	5	9	9	2	9	9	9
$j=2$	3	4	7	3	6	5	9	9
$j=1$	2	3	5	0	5	9	9	9
$j=0$	1	2	3	5	0	5	9	9
	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$

The i = 6 column has only 9s, so that will hold for all subsequent columns. We call n "stable" if the columns of B*[n] are eventually constant. Let s(n) be the column index of the first constant column. So s(13) is 6. If the columns never become constant, then s(n) = ∞. Here are some s-values, where n starts at 1 and ends at 13.

0, 2, 4, 3, 5, 4, 6, 3, 6, 5, 6, 4, 6

Part (a) asks whether the columns always become constant. Part (b) asks if the number 4 occurs infinitely often as s(n).