# A possible proof of the Powers of Two problem

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This document is a response to the video entitled: `Problems with Powers of Two - Numberphile` that appeared on Youtube on September 21st, 2022 (submitted by Numberphile).

Video id.: `https://youtu.be/IPoh5C9CcI8`

The author of this document intends to provide a sketch of a possible proof of the **four numbers case of the powers of two problem** detailed at the beginning of the outlined video. The author of this document believes that the ideas presented here can be used to formulate rigorous mathematical proof for this particular case and are helpful for generalizations of the problem to higher dimensions (number of integers).

## 1 Problem statement

Let $\mathbb{Z}$ denote the set of integers. We want to show that there exist $z_1 \neq z_2 \neq z_3 \neq z_4 \in \mathbb{Z}$ and $s_i > 0$, $\log_2(s_i) \in \mathbb{Z}$ $(i = 1, \ldots, 6)$ numbers so that

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix}, \quad \text{or} \quad \mathbf{Az} = \mathbf{s} \quad \text{(in short)}. \tag{1}$$

## 2 Sketch of the proof

Let us assume that there exists a solution to the outlined problem. In our scenario, the necessary condition for the existence of a solution is: $\mathbf{s} \in \mathrm{Im}(\mathbf{A})$ where $\mathrm{Im}(\mathbf{A})$ denotes the range space (column space) of matrix $\mathbf{A}$. In other words, vector $\mathbf{s}$ must be in the linear subspace spanned by the column vectors of $\mathbf{A}$. An equivalent statement: vector $\mathbf{s}$ must be perpendicular to the orthogonal complement of $\mathrm{Im}(\mathbf{A})$ defined by the nullspace of $\mathbf{A}^T$ denoted by $\mathrm{Ker}(\mathbf{A}^T)$. It is easy to verify that $\mathrm{Ker}(\mathbf{A}^T) = \mathrm{Im}(\mathbf{N})$ where (for example)

$$\mathbf{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

since $\mathbf{A}^T\mathbf{N} = \mathbf{0}$. (It is also easy to verify that $\mathrm{Im}([\mathbf{A}\ \mathbf{N}]) = \mathbb{R}^6$). Consequently,

$$\mathbf{s} \in \mathrm{Im}(\mathbf{A}) \quad \Rightarrow \quad \mathbf{N}^T\mathbf{s} = \mathbf{0} \quad \rightarrow \quad \begin{array}{rcl} s_1 - s_3 - s_4 + s_6 & = & 0 \\ s_2 - s_3 - s_4 + s_5 & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} s_1 + s_6 & = & s_3 + s_4 \\ s_2 + s_5 & = & s_3 + s_4 \end{array} \tag{3}$$

Using binary representation for $s_i$ (e.g., $s_1 = 8 = \langle \ldots 001000.000 \ldots \rangle_2$ or in case we allow negative powers $0.125 = \langle \ldots 0000.00100 \ldots \rangle_2$ ) it is easy to see that: given equality (3), $s_1, \ldots, s_6$ can not be distinct since any binary sequence including exactly two 1-bits (sum of distinct integer powers of two) can be uniquely decomposed to a sum of two binary sequences, each including exactly one 1-bit (integer powers of two). Furthermore, given that $s_i > 0$, the possible solutions to (3) are:

$$\text{1st equation:} \begin{cases} s_1 = s_3 & \Rightarrow \quad s_4 = s_6 \\ s_1 = s_4 & \Rightarrow \quad s_3 = s_6 \end{cases} \qquad \text{2nd equation:} \begin{cases} s_2 = s_3 & \Rightarrow \quad s_4 = s_5 \\ s_2 = s_4 & \Rightarrow \quad s_3 = s_5 \end{cases}$$

giving us five possible ways to construct vector $\mathbf{s}$:

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} x & x & x & x & x \\ x & x & y & y & x \\ x & x & x & y & y \\ x & y & y & x & x \\ x & y & x & x & y \\ x & y & y & y & y \end{array}\right] \tag{4}$$

where $x, y > 0$ and $\log_2(x) \in \mathbb{Z}$, $\log_2(y) \in \mathbb{Z}$ ($x, y$ are integer powers of 2). Now, taking the first four rows of $\mathbf{A}$ and $\mathbf{s}$ and performing Gaussian elimination, we get the following equality:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 - s_1 \\ s_3 - s_1 \\ s_4 + s_2 - s_1 \end{bmatrix}$$

with solutions in the following form:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} s_1 - z_2 \\ z_3 + s_1 - s_2 \\ z_3 \\ z_2 + s_3 - s_1 \end{bmatrix} = \begin{bmatrix} -z_3 + s_2 \\ z_3 + s_1 - s_2 \\ z_3 \\ z_3 + s_3 - s_2 \end{bmatrix}. \tag{5}$$

Putting (5) together with (4) we get the following table:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} -z_3 + x & -z_3 + x & -z_3 + y & -z_3 + y & -z_3 + x \\ z_3 & z_3 & z_3 + x - y & z_3 + x - y & z_3 \\ z_3 & z_3 & z_3 & z_3 & z_3 \\ z_3 & z_3 & z_3 + x - y & z_3 & z_3 + y - x \end{array}\right].$$

As can be seen, each possible result has repetitive values implying that there exist no $z_1 \neq z_2 \neq z_3 \neq z_4 \in \mathbb{Z}$ and $s_i > 0$, $\log_2(s_i) \in \mathbb{Z}$ $(i = 1, \ldots, 6)$ subject to (1).