Stan Wagon 1321 Solution

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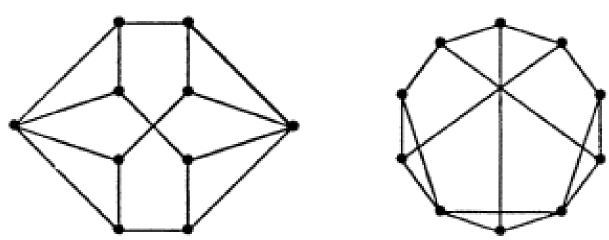
Stan Wagon's Problem of the week #1321 asks us to find 10 distinct integers such that at least 14 of their pairwise sums are powers of 2. 15 is possible, given by $\{-5, -3, -1, 1, 3, 5, 7, 9, 11, 13\}$, and an upper bound of 16 is known. In fact, 10 is the smallest number for which the true value is unknown. In this paper we show that 15 is indeed optimal.

1 Introduction

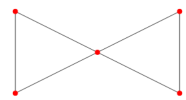
Our main method of attack will be based on an idea of M.S.Smith. We will draw a graph, henceforth called a "power of 2 graph" on our 10 integers, connecting two of them if and only if their sum is a power of 2. Smith was able to show the graph contains no subgraph isomorphic to C_4, which is already enough to lower the bound on the number of edges to 16. In fact, there are only two possibilities for a 16 edge 10 vertex squarefree graph by work of Clapham et. al [1], which

Figure 1: The two graphs, labelled A and B from left to right, which we must eliminate



are shown in Figure 1. We will show neither of these possibilities can occur, by showing that any set of 5 numbers for which the power of 2 graph is the butterfly graph (See Figure 2) must have a very special form. From this we are

Figure 2: The 5-vertex butterfly graph



able to show neither of the two square free 10 vertex graphs with 16 edges can occur.

2 Special Subgraphs

Given a set of k integers $\{a_1, \dots a_k\}$, form a graph by connecting a_i and a_j if $a_i + a_j$ is a power of 2. We call any subgraph of a graph formed in this manner a *power of 2 graph*. These graphs have very restricted structure, as one sees from the following theorem originally due to Smith:

Theorem 1. *Power of* 2 *graphs are "squarefree" in that they contain no subgraph isomorphic to* C_4.

Proof. Such a subgraph corresponds to a solution to the system of equations

$$\begin{aligned} a + b &= 2^{n_1} \\ b + c &= 2^{n_2} \\ c + d &= 2^{n_3} \\ d + a &= 2^{n_4} \end{aligned}$$

with the n_i non-negative integers and a, b, c, d distinct integers. Then

$$2^{n_4} - 2^{n_3} = 2^{n_1} - 2^{n_2}.$$

If now $n_4 = n_3$, we find $c = a$, a contradiction. A similar argument shows $n_2 \neq n_1$. By considering the sign and 2-adic valuation of both sides, we see that $n_2 = n_3$ is forced, but this implies $b = d$, a contradiction. Thus any such graph cannot contain a subgraph isomorphic to C_4. □

This theorem is already enough to yield very good bounds on its own. As an example, there is a classical argument that in a squarefree graph with n vertices there are not more than $\frac{n}{4}(1 + \sqrt{4n-3})$ edges.

In order to explicitly solve the case of $n = 10$ vertices, we state the following result of Clapham et. al.

Theorem 2. *A square free graph with* 10 *vertices has at most* 16 *edges, and there are exactly* 2 *such graphs with* 16 *edges (See figure 1).*

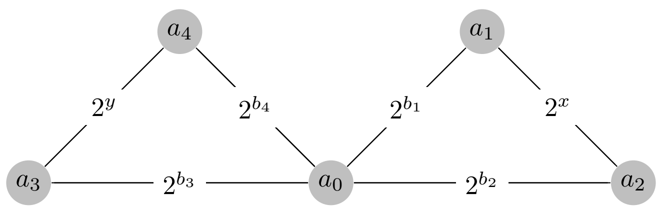
Proof. See [1] □

We now state our main result.

Theorem 3. *Neither of the two squarefree graphs with* 10 *vertices and* 16 *edges is a power of 2 graph of some set of integers. Therefore,* 15 *is indeed the optimal solution to problem of the week 1321.*

The proof of Theorem 3 will rely heavily on the following lemma:

Lemma 4. *Let a_0, a_1, a_2, a_3, a_4 be distinct integers whose associated power of* 2 *graph is the butterfly graph, with a_0 the vertex of degree* 4 *and $\{a_0, a_1, a_2\}$ one of the* 3*-cycles. Then there exist integers $n, m \geq 0$ with $n \geq m+2$ such that, up to a graph automorphism, $a_0 = 2^n + 2^m$, $a_1 = 2^m - 2^n$, $a_2 = 3(2^n) - 2^m$, $a_3 = 3(2^m) - 2^n$, $a_4 = 2^n - 2^m$. In particular, a_1 and a_3 are negative and the same power of* 2 *divides every a_i.*



Proof. It will be useful to first argue which powers of 2 are possible sums for all of the edges. Therefore let $2^x = a_1 + a_2$, $2^y = a_3 + a_4$ be the powers of 2 associated to the "wings" of the butterfly. Similarly let $2^{b_i} = a_0 + a_i$ for $i \in \{1, 2, 3, 4\}$. If two edges are joined to the same vertex and have the same value, the values of the other vertices they connect to must be the same. Thus distinctness of the a_i implies distinctness of the b_i. Similarly we have that x, b_1, b_2 are distinct and that y, b_3, b_4 are distinct.

The various definitions now give that

$$2^{b_1} - 2^x + 2^{b_2} = 2a_0 = 2^{b_3} - 2^y + 2^{b_4}$$

Distinctness of the powers of 2 occurring on each side tells us that $\min\{x, b_1, b_2\} = \min\{y, b_3, b_4\}$. This together with distinctness of the b_i tells us that up to a graph automorphism that either $x = y = \min\{x, b_1, b_2\}$ or $b_1 = y = \min\{x, b_1, b_2\}$. If the first case holds, then $2^{b_1} + 2^{b_2} = 2^{b_3} + 2^{b_4}$, which quickly contradicts the idea that the b_i are distinct. Therefore we assume $b_1 = y = \min\{x, b_1, b_2\}$. Thus,

$$2^{b_1+1} - 2^x + 2^{b_2} = 2^{b_3} + 2^{b_4}.$$

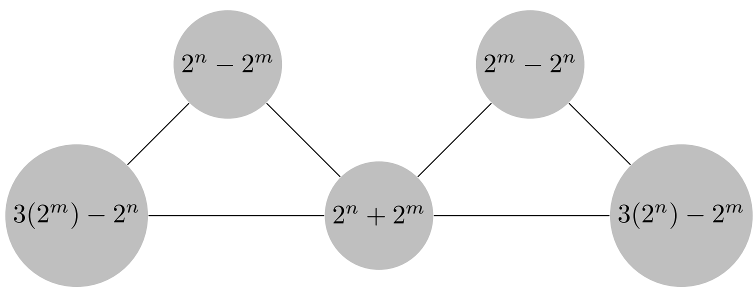
Positivity of the right hand side and the fact $b_1 = \min\{x, b_1, b_2\}$ tells us $b_1 + 1 \leq x < b_2$. If $b_1 + 1 = x$, then $2^{b_2} = 2^{b_3} + 2^{b_4}$, contradicting the idea $b_3 \neq b_4$. Thus 2^{b_1+1} is the largest power of 2 dividing the left hand side. As $b_3 \neq b_4$, up to a graph automorphism (which will always be compatible with our first graph automorphism) we can assume $b_1 + 1 = b_3$. We then see $2^{b_2} - 2^x = 2^{b_4}$, which forces $x = b_2 - 1$. Thus, letting $b_1 = m + 1$, $b_2 = n$, we have shown that

$$\begin{aligned} a_0 + a_1 &= 2^{b_1} = 2^{m+1} \\ a_0 + a_2 &= 2^{b_2} = 2^{n+2} \\ a_0 + a_3 &= 2^{b_3} = 2^{m+2} \\ a_0 + a_4 &= 2^{b_4} = 2^{n+1} \\ a_1 + a_2 &= 2^{x} \ \ = 2^{n+1} \\ a_3 + a_4 &= 2^{y} \ \ = 2^{m+1} \end{aligned}$$

Solving this system yields the unique solution that

$$\begin{aligned} a_0 &= 2^n + 2^m \\ a_1 &= 2^m - 2^n \\ a_2 &= 3(2^n) - 2^m \\ a_3 &= 3(2^m) - 2^n \\ a_4 &= 2^n - 2^m \end{aligned}$$

as desired.



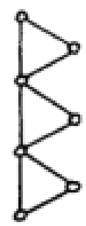
Note m, n were such that $m = b_1 - 1 \leq b_2 = n$, and that distinctness of the a_i fails if $m = n$ or $m + 1 = n$, so $m - n \geq 2$. This quickly implies $a_1, a_3 < 0$. □

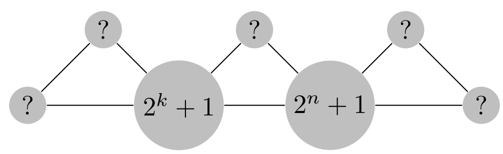
We immediately use this lemma to eliminate possibilities of certain graphs occurring as subgraphs of a power of 2 graph.

Corollary 4.1. *The 7 vertex graph T consisting of 3 triangles joined as in figure 3 is not a power of 2 graph.*

Proof. This graph contains two copies of the butterfly graph which share vertices, so the same power of 2 will divide every vertex. Thus by Lemma 4 we can cancel a common power of 2 and assume the center of one butterfly is of the form $2^n + 1$ and the other is of the form $2^k + 1$, $n \neq k$ and that both $n, k > 1$, so the situation is as below.

Figure 3: This 7 vertex graph T does not occur as a power of 2 graph





As $2^k + 1$ is positive, by Lemma 4 we must have either $2^k + 1 = 2^n - 1$ or $2^k + 1 = 3(2^n) - 1$, but both equations fail modulo 4 since $k, n > 1$. Thus T is not a power of 2 graph. □

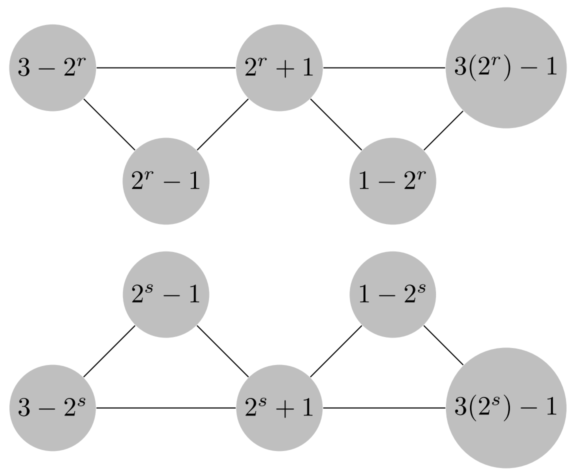
Corollary 4.2. *The 16 edge 10 vertex graph B in figure 1 is not a power of 2 graph.*

Proof. It contains the graph T of Corollary 4.1 as a subgraph, and thus cannot appear. □

Corollary 4.3. *The 16 edge 10 vertex graph A in figure 1 is not a power of 2 graph*

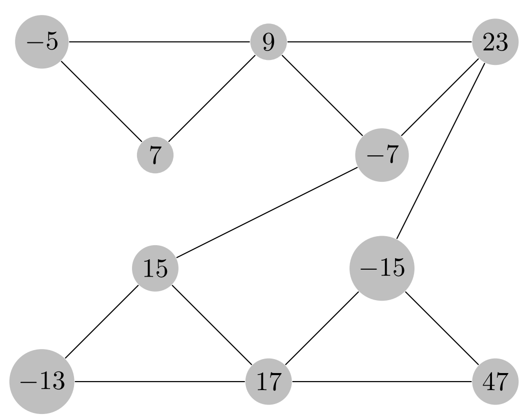
Proof. This graph contains two butterfly graphs as subgraphs, joined by 4 edges attached to the wing vertices of the butterflies. We first aim to show that we can assume every vertex is an odd number. To do this, suppose by Lemma 4 one butterfly has central vertex $2^a + 2^b$ and the other has central vertex $2^c + 2^d$. By cancelling a common power of 2 suppose without loss of generality that $d = 0$. In particular, one butterfly now consists of only odd numbers. If the other butterfly consists only of even numbers, the sum along the 4 edges joining the two butterflies must be 1, the only odd power of 2. Then the sum of the 8 vertices of these edges must be 4, but by Lemma 4 this sum will also equal $2^{a+1} + 2^{b+1} + 2^{c+1} + 2$. Since one of $a + 1$ or $b + 1$ is greater than 2, $2^{a+1} + 2^{b+1} + 2^{c+1} + 2 > 4$, a contradiction. Thus, after cancelling a common power of 2, we can assume the vertices of our graph are only odd numbers.

Now suppose the centers of our butterflies are $2^r + 1$ and $2^s + 1$.



Since we cannot join a negative vertex to a negative vertex and hope for them to sum to a positive power of 2, we need only consider ways of joining the butterflies which connect negative vertices to positive ones. It turns out there are exactly two ways to join up the two butterflies in a way such that the resulting graph is isomorphic to A and that every negative vertex is joined to a positive one. In the first, we get the condition that the 4 edge sums $3(2^r) - 2^s$, $2^s - 2^r$, $3(2^s) - 2^r + 2$, $2^r - 2^s + 2$ must be powers of 2, but Lemma 4 implies r and s are both greater than 1, so $2^r - 2^s + 2$ is not a square modulo 4. The analysis in the second case is identical with the roles of r and s swapped, so indeed A is not a power of 2 graph. □

Now corollaries 4.2 and 4.3 complete the proof of Theorem 3, so we have indeed established that the known set of 15 is optimal. We note that our proof of Corollary 4.3 yields infinitely many easy sets of 14 containing only odd numbers, as taking $s = r + 1$ makes half of the sums joining the butterflies work. The case $r = 3$ is illustrated



References

[1] Clapham, C. R. J., A. Flockhart, and J. Sheehan. “Graphs without Four-cycles.” Journal of Graph Theory 13, no. 1 (March 0, 1989): 29–47. doi:10.1002/jgt.3190130107.