FAIRLY 3-REGULAR GRAPHS A352175

RICHARD J. MATHAR

Abstract. These are illustrations of the 1, 5, 26, 146 unlabeled undirected fairly cubic graphs on 2, 4, 6, 8 nodes closely connected to the first 4 terms of sequence A352175 of the OEIS.

1. Introduction

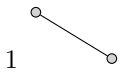
Fairly cubic graphs on n nodes are graphs with 2 nodes of degree 1 and $n-2$ nodes of degree 3—nomenclature as in [5, 1]. (In some parts of the literature these are enumerated as graphs on $n-2$ nodes with two free legs, and occasionally these two legs are called fins (half-edges) and count as one node [2].) Alternatively one may remove the nodes of degree 1 and their incident edges and consider graphs with 2 nodes of degree 2 (formerly adjacent to the leafs) and $n-2$ nodes of degree 3 which obviously leafs all information intact. This text illustrates graphs which may have loops and multi-edges and any number of components.

A result of the handshake lemma is that graphs with odd n do not exist; only the cases of even n are relevant.

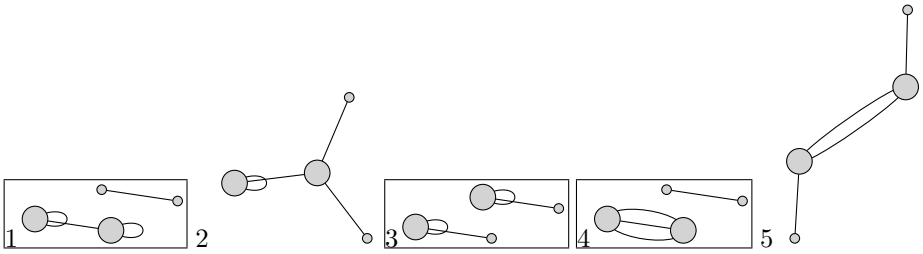
The two nodes with degree 1 (leaves) are plotted with smaller radius than the nodes of degree 3. Graphs with more than one component are boxed for visual clarity. Graphs are enumerated at the lower left edge starting at 1 (...sometimes this index may appear at the end of previous line).

Extension of the concept to digraphs is possible [4].

2. 1 graph on 2 nodes



3. 5 graphs on 4 nodes (2 connected)



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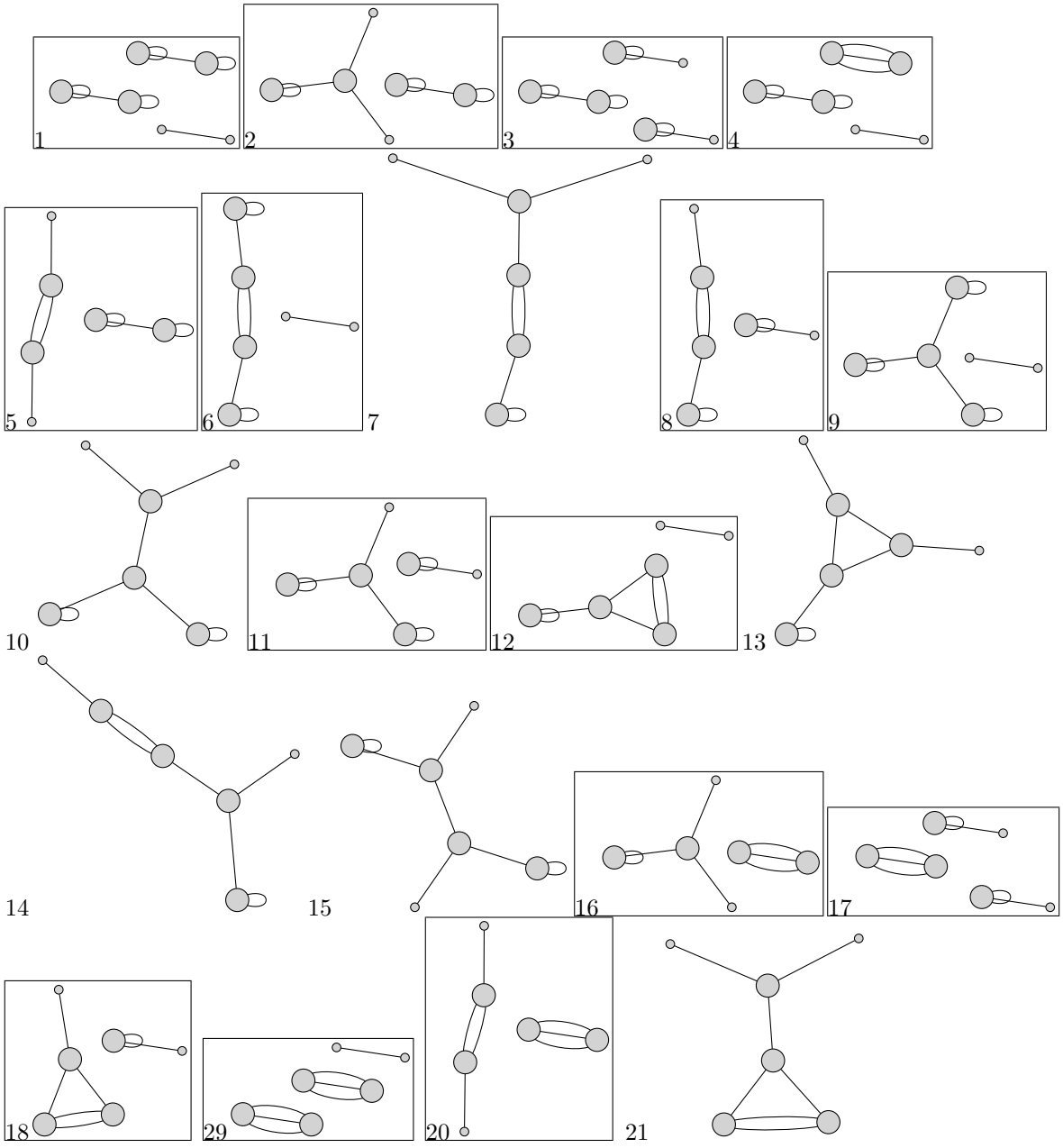
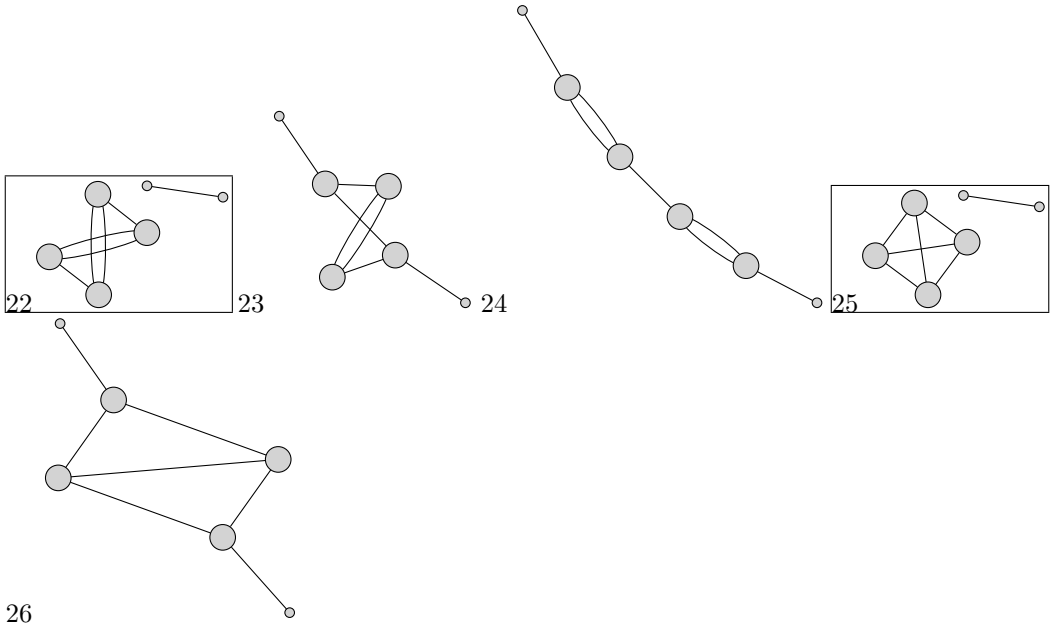
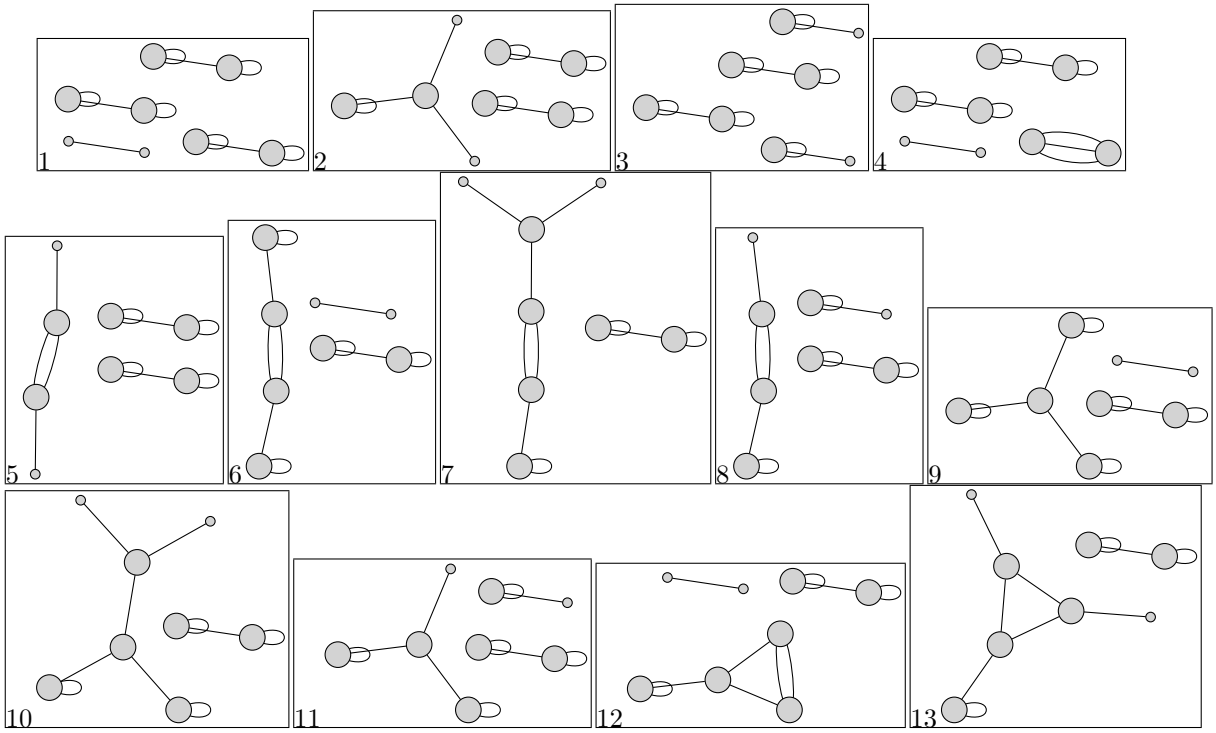
Key words and phrases. Graph Enumeration, Cubic Graphs, phi(3).

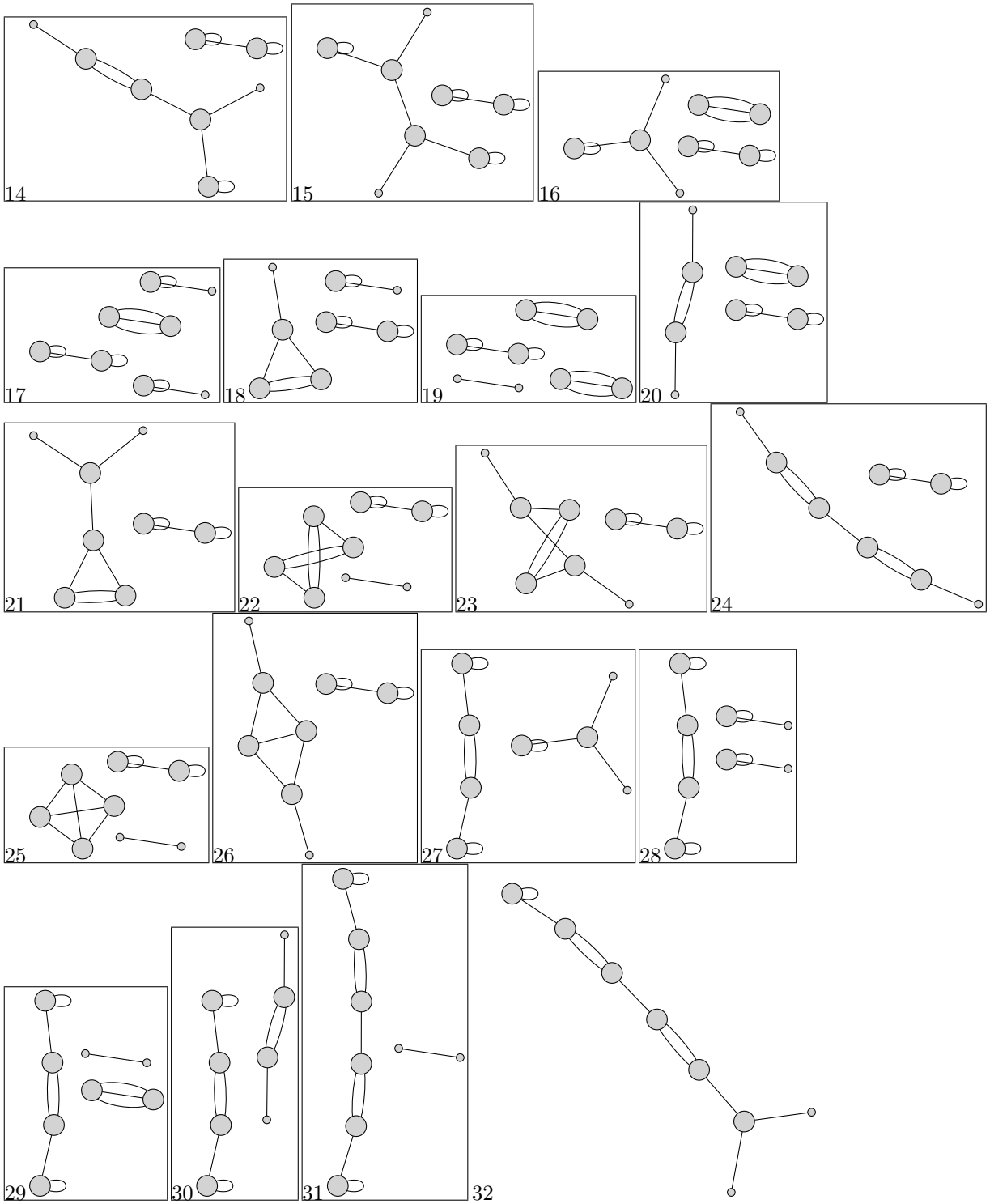
4. 26 graphs on 6 nodes (9 connected)





5. 146 graphs on 8 nodes (49 connected)

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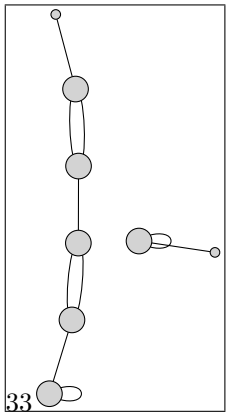
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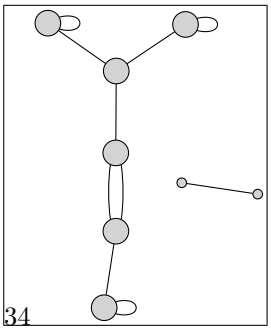
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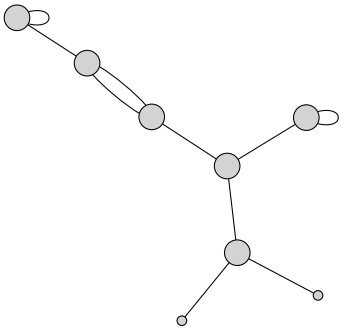
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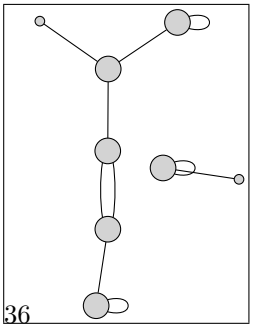
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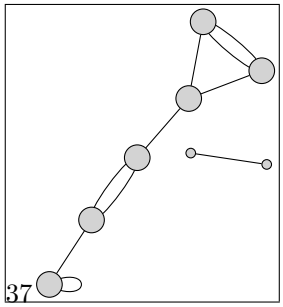
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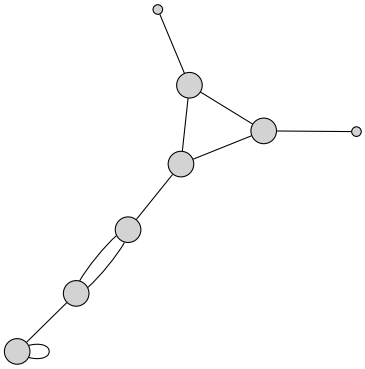
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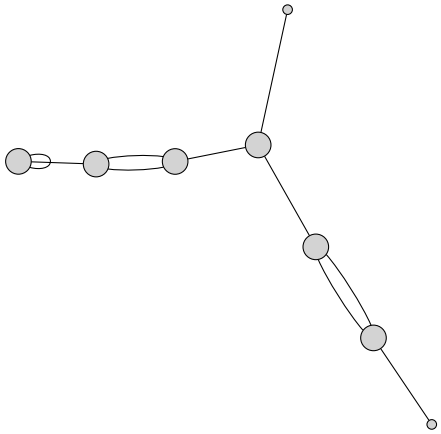
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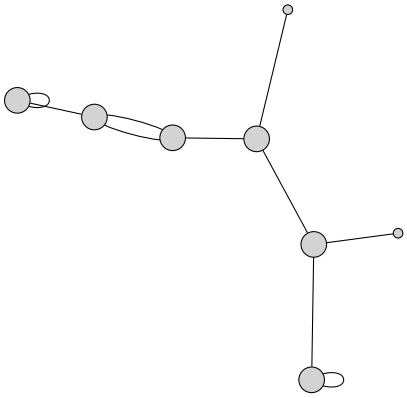
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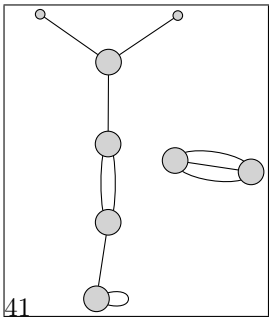
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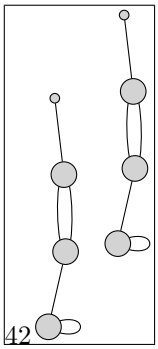
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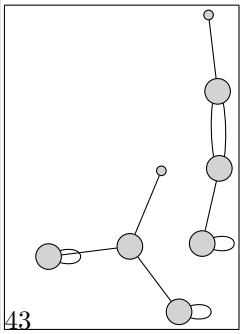
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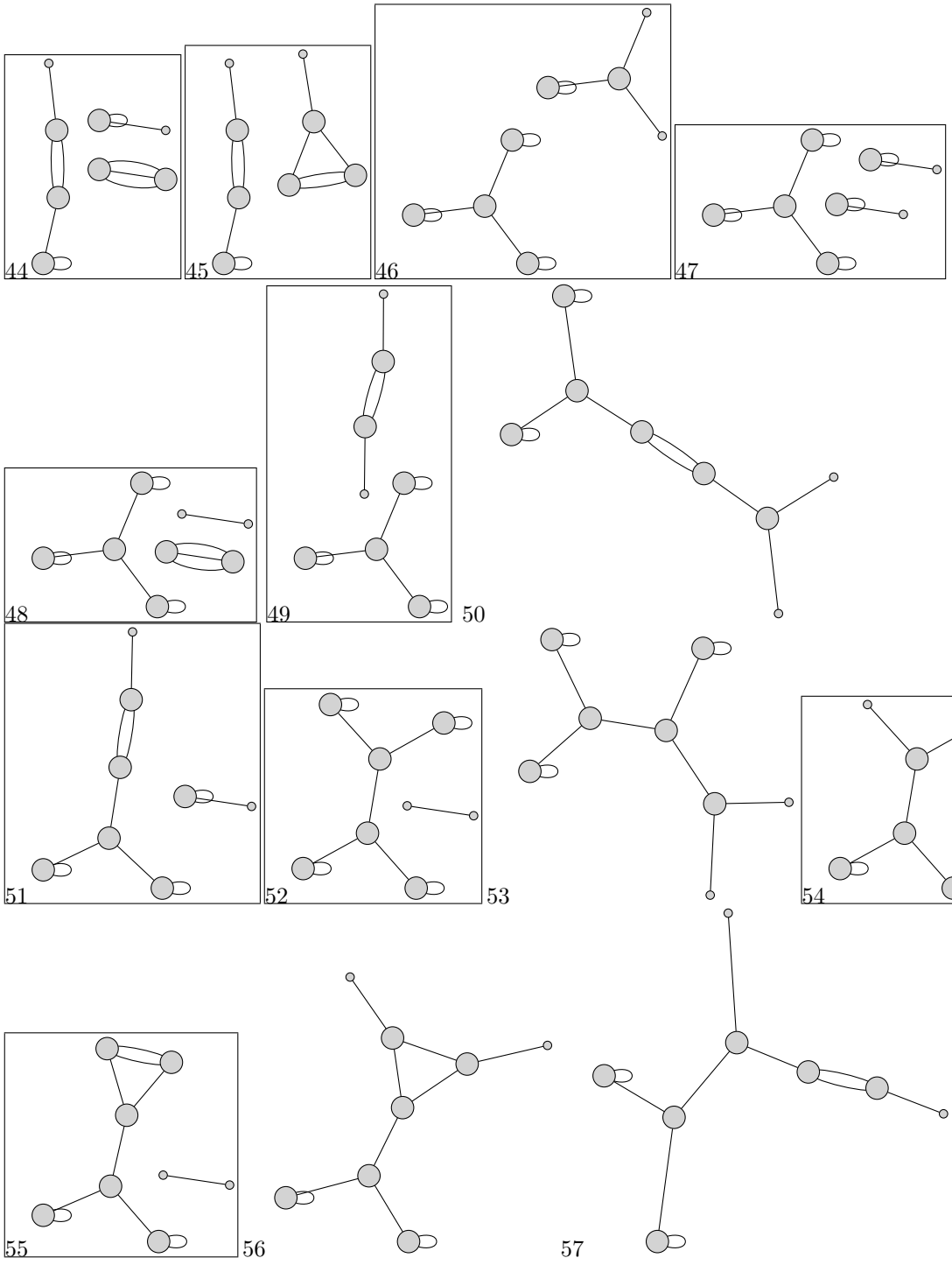
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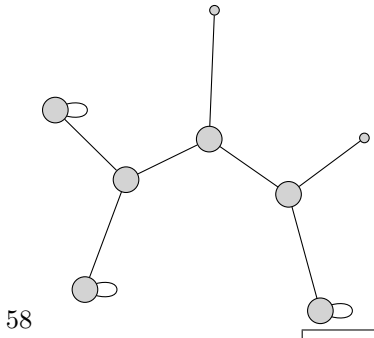
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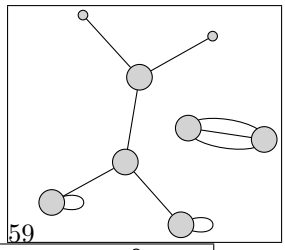
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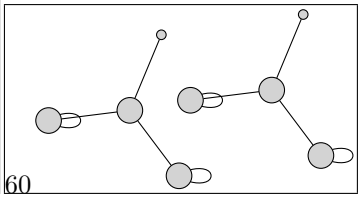
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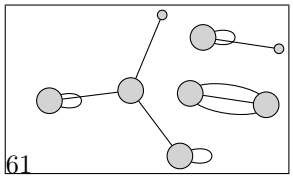
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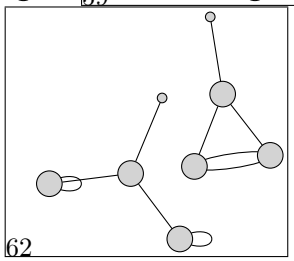
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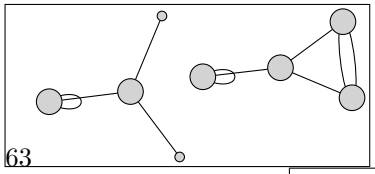
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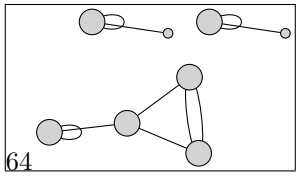
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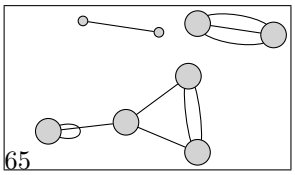
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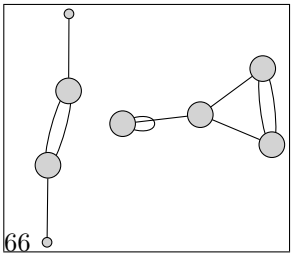
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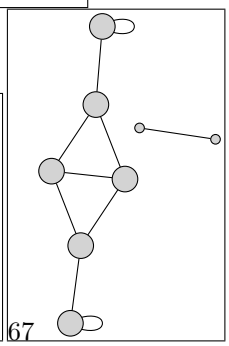
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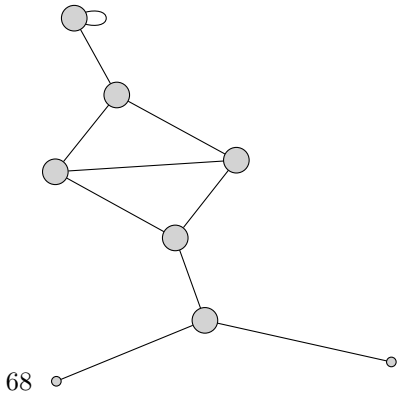
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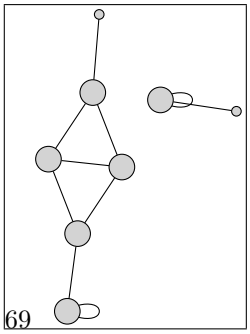
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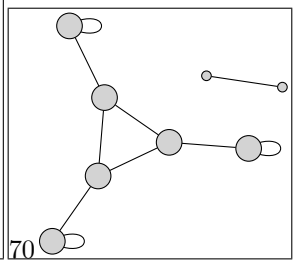
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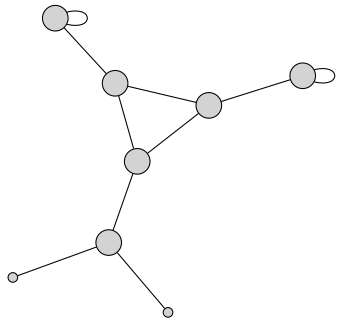

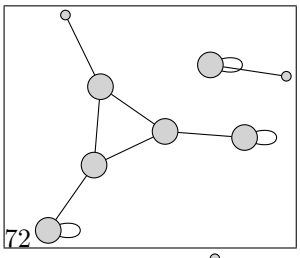

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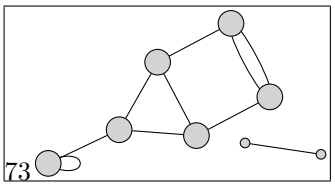

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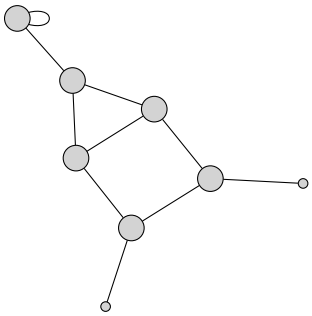

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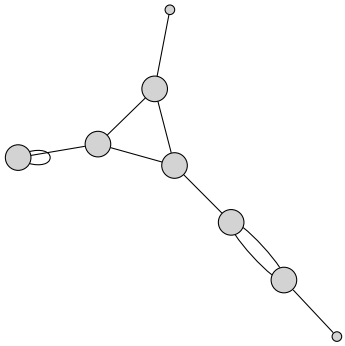

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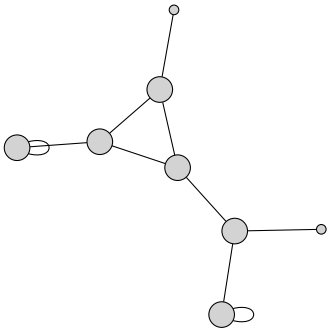

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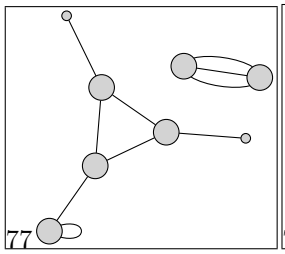

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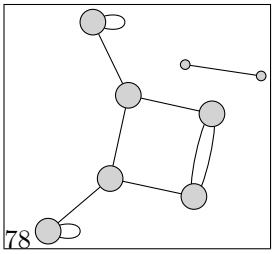

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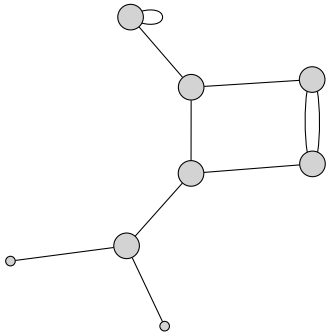

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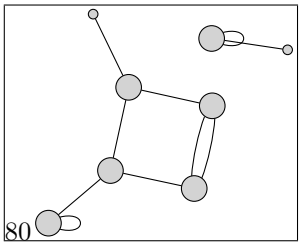

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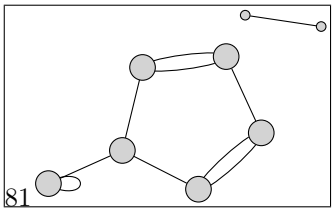

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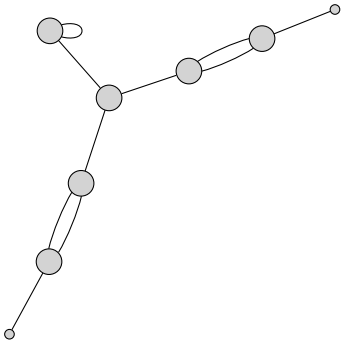

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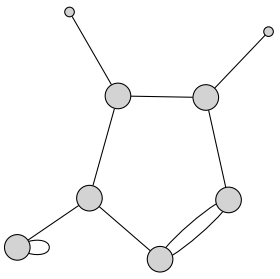

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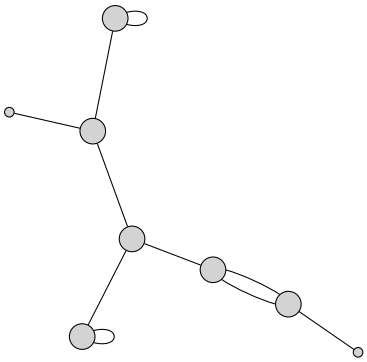

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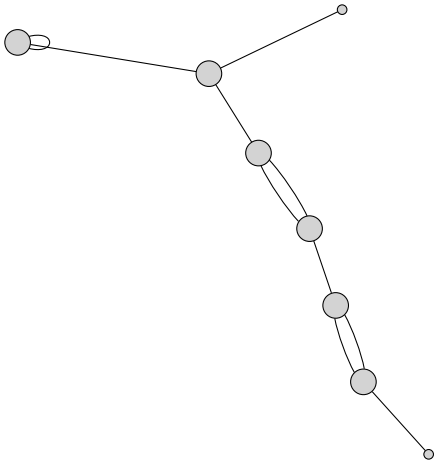

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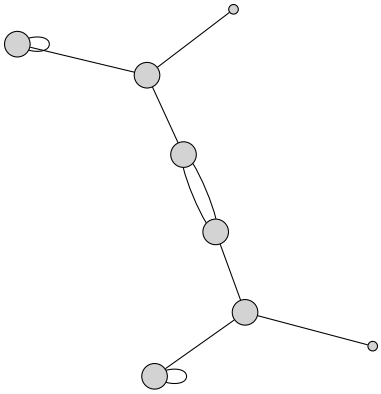

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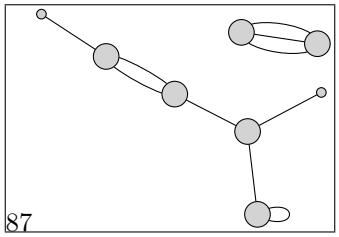

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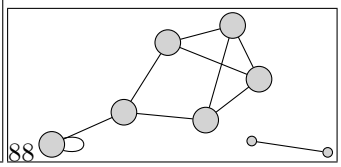

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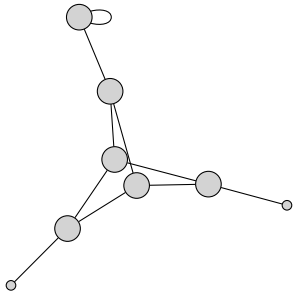

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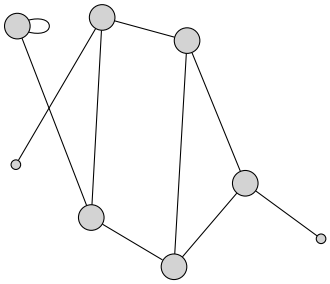

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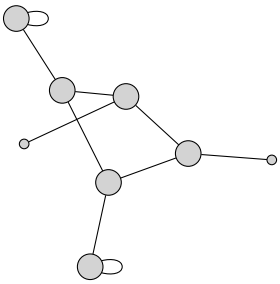

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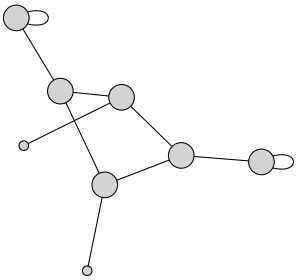

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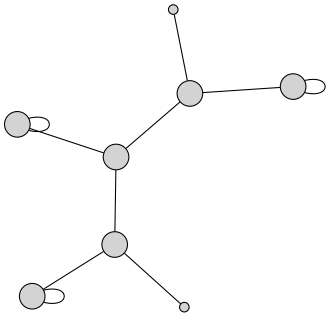

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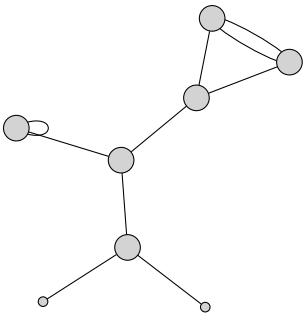

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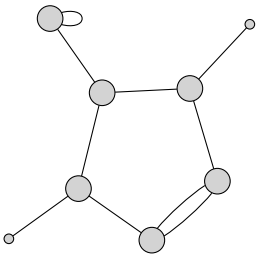

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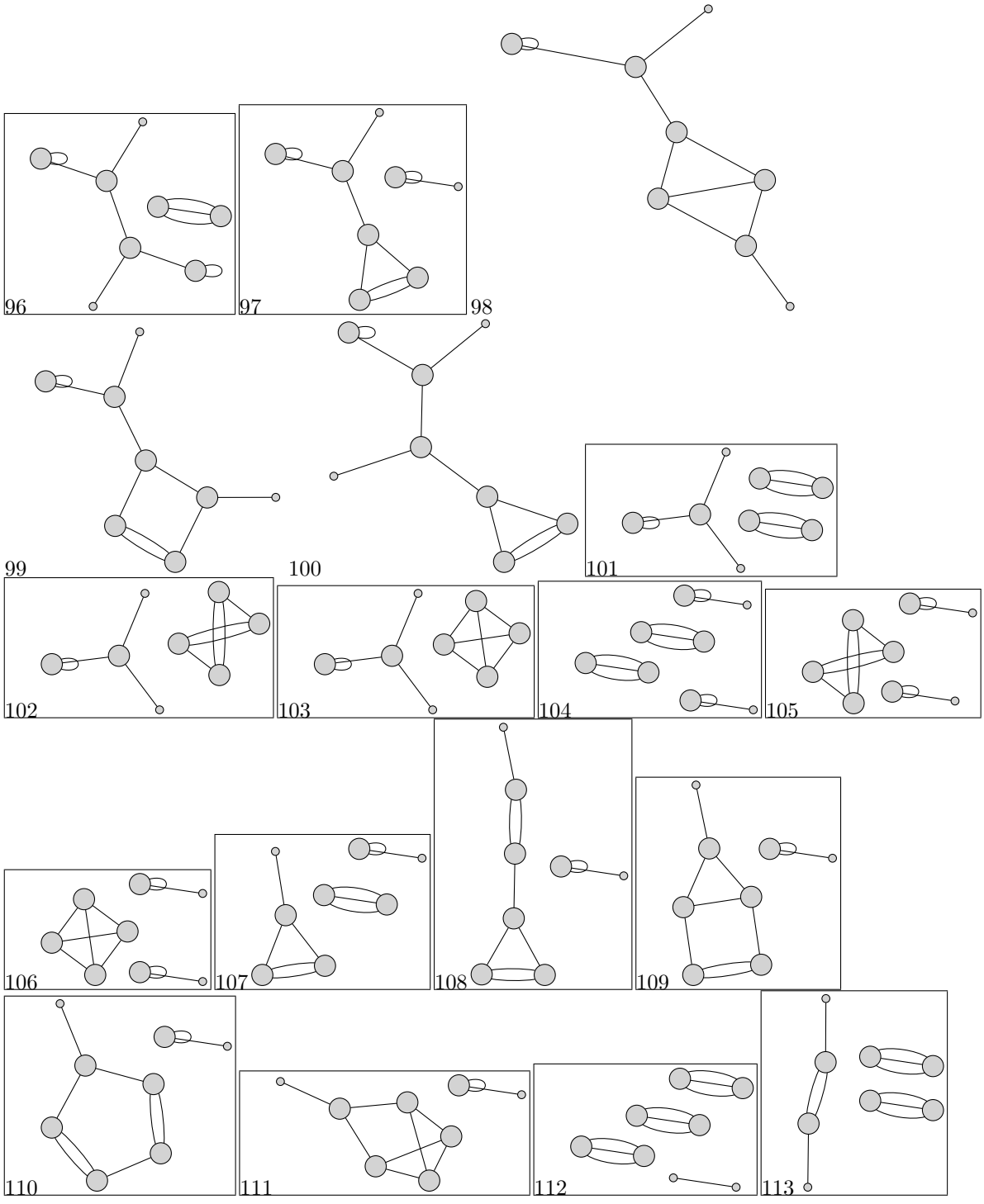
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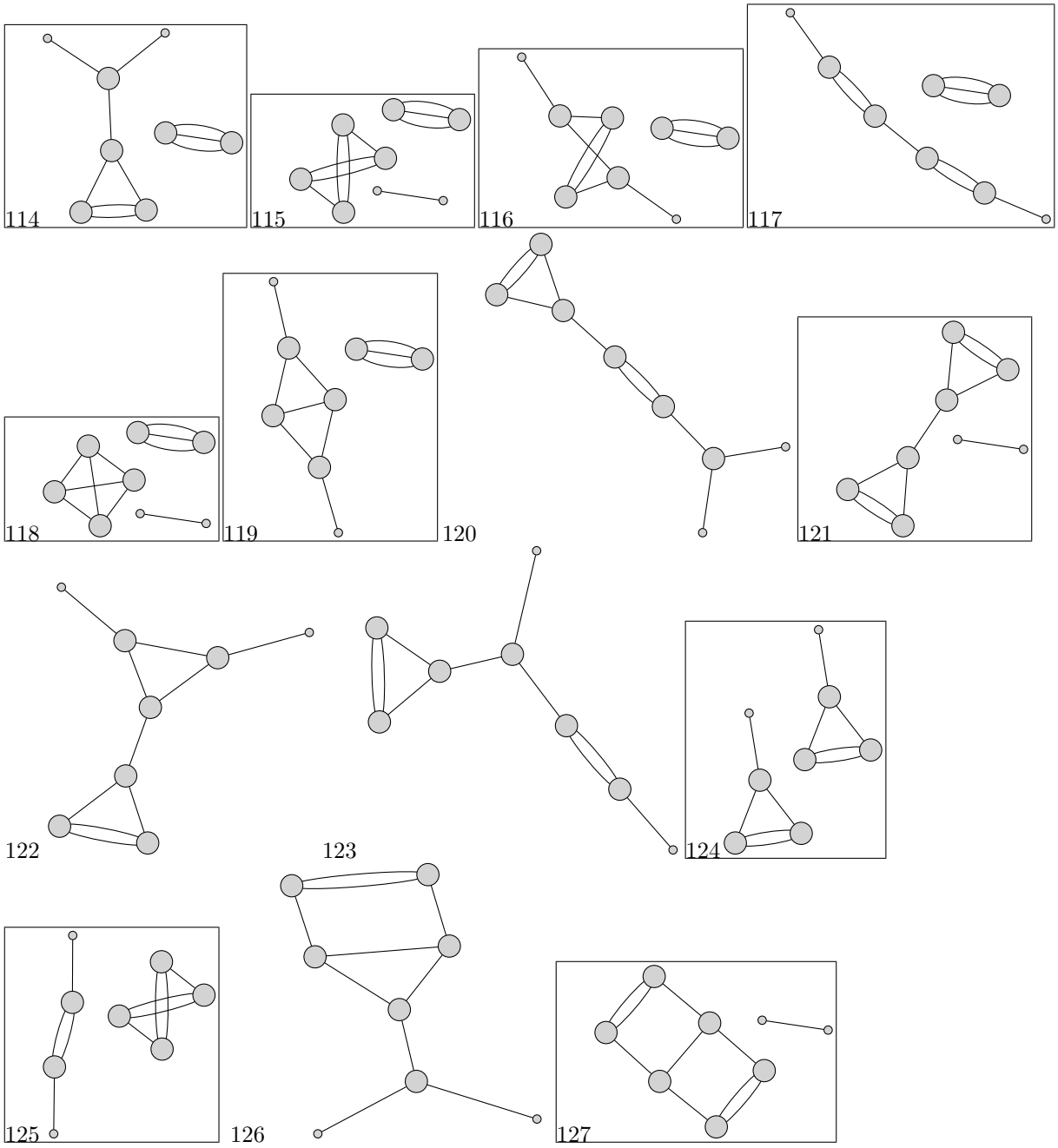
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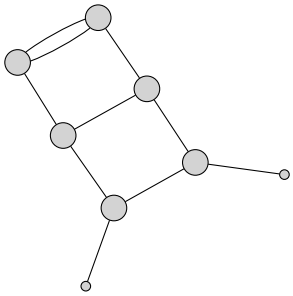
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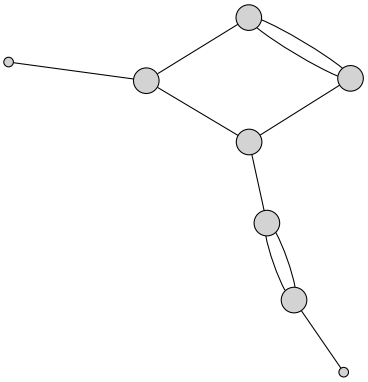
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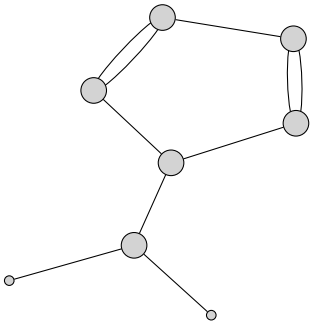
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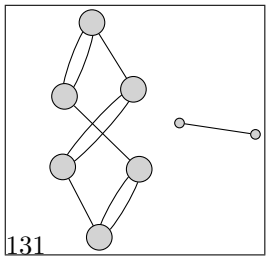
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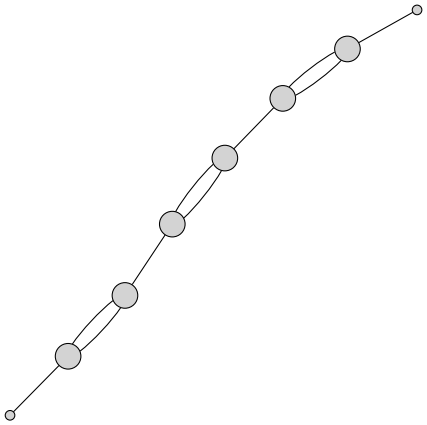
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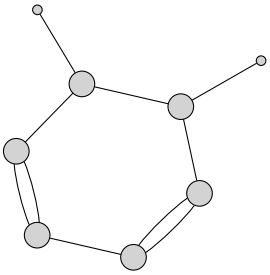
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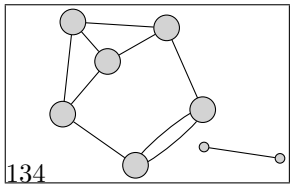
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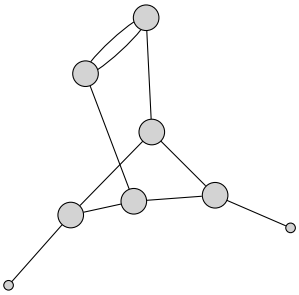
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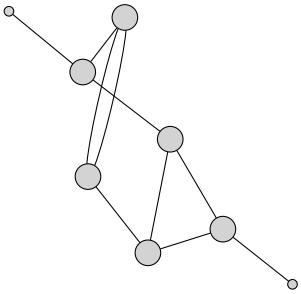
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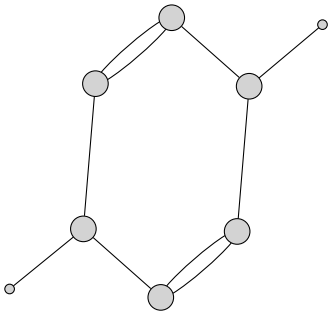
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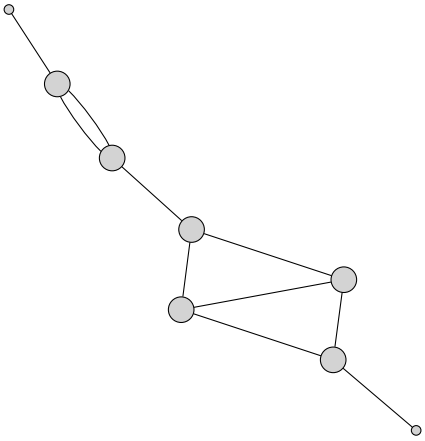
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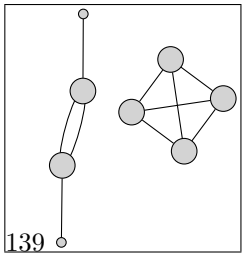
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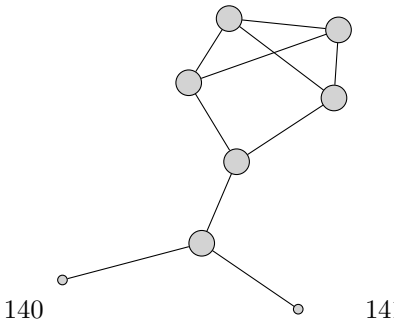
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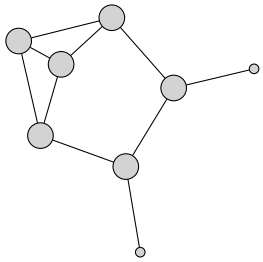
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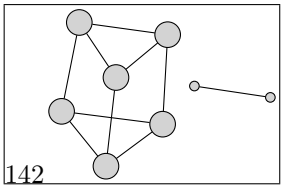
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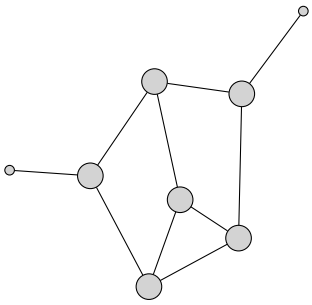
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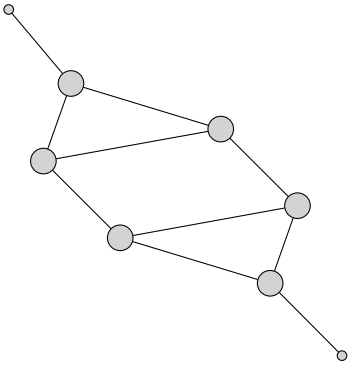
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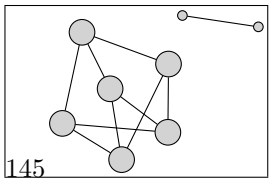
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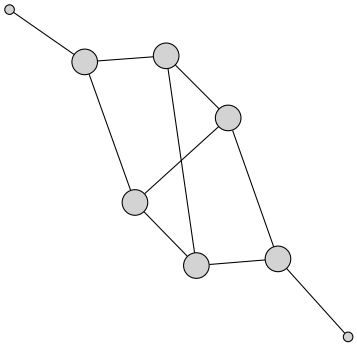
146

6. Discussion

6.1. **Marked/rooted leafs.** The enumeration by de Mello Koch and Ramgoolam [2][3, A352175] is obtained by considering one of the two nodes of degree 1 marked (or both labeled 1 and 2) such that the graphs which are not symmetric with respect to these two nodes are counted twice. Alternatively one might say they count the fairly cubic graphs rooted on the leafs.

(1) On 4 nodes no graph has this asymmetry.
(2) On 6 nodes graphs number 8, 11 14 and 18 have this lower symmetry so these authors count $26 + 4 = 30$ relevant graphs.
(3) On 8 nodes graphs number 8, 11, 14, 18, 33, 36, 39, 40, 43, 44, 45, 51, 54, 57, 58, 61, 62, 69, 75, 76, 80, 83, 84, 85, 87, 90, 97, 98, 99, 100, 107, 108, 109, 110, 111, 123, 129, 135, 136 and 138 have this lower symmetry, so these authors count $146 + 40 = 186$ here.

6.2. **Connected vs. disconnected graphs.** The fact that the two leafs may be nodes in two different components of a graph leads to a slightly more convoluted enumeration of general graphs given the enumeration of the connected graphs relative to the "standard" case (where the generating function were the generating function of the cubic graphs—without leafs as in [3, A129427]—multiplied by the generating function of the connected graphs, essentially taking one fairly cubic graph and putting any number of cubic graphs into further components). The graphs are classified as

- one connected fairly cubic graph plus any number of cubic graphs, or
- two graphs with a single leaf node plus any number of cubic graphs.

The generating function (GF) for the cubic graphs of any number of components (allowing multi-edges and loops) is [3, A129427]

$$C(x) = 1 + 2x^2 + 8x^4 + 31x^6 + 140x^8 + \cdots \tag{1}$$

Remark 1. *The 1, 2, 8 and 31 graphs are illustrated by those graphs in Sections 2–4 where the simple graph on 2 nodes is a component and that component then deleted.*

The GF for the fairly cubic graphs illustrated above is

$$F(x) = 1 + x^2 + 5x^4 + 26x^6 + 146x^8 + \cdots \tag{2}$$

The GF for the subset of *connected* fairly cubic graphs illustrated above is

$$F^{(c)}(x) = 1 + x^2 + 2x^4 + 9x^6 + 49x^8 \cdots \tag{3}$$

The GF for the connected almost cubic graphs—with 1 node of degree 1 and all other nodes of degree 3—is

$$A^{(c)}(x) = 1 + x^2 + 3x^4 + 12x^6 + 67x^8 \cdots \tag{4}$$

See e.g. graphs 11 on 6 nodes or 107 or 111 on 8 nodes where two of the almost cubic graphs with 2, 4 or 6 nodes appear as components.

Remark 2. *The GF for a cubic graphs with 2 components which are almost-cubic (where components may be empty) is by the usual counting argument involving the automorphisms of the symmetric group S_2*

$$\hat{A}(x) = [A^{(c)}(x^2) + {A^{(c)}}^2(x)]/2 = 1 + x^2 + 4x^4 + 15x^6 + 85x^8 + \cdots \tag{5}$$

The GF for a cubic graphs with 2 components which are almost-cubic (and no component is empty) is given by

$$\bar{A}(x) = [A^{(+c)}(x^2) + {A^{(+c)}}^2(x)]/2 = x^4 + 3x^6 + 18x^8 + \cdots \tag{6}$$

where $A^{(+c)}(x) \equiv A^{(c)}(x) - 1$ are the nonempty connected almost cubic graphs.

Example 1. *The one graph on 4 nodes is graph 3 in Section 3; the 3 graphs on 6 nodes are graph 8, 11, and 18 in Section 4; The 18 graphs on 8 nodes are graphs 33, 36, 42, 43, 45, 51, 54, 60, 62, 69, 72, 80, 97, 108, 109, 110, 111, and 124 in Section 5.*

The GF $F^{(c)}(x) + \bar{A}(x) - 1$ counts the nonempty fairly cubic graphs where the leafs are in the same or two different components. Finally

$$F(x) = C(x)[F^{(c)}(x) + \bar{A}(x) - 1] = x^2 + 5x^4 + 26x^6 + 146x^8 + \cdots \tag{7}$$

relates the fairly cubic graphs to the cubic graphs, the connected fairly cubic graphs and the almost cubic graphs.

References

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2. Robert de Mello Koch and Sanjaye Ramgoolam, *Strings from feynman grpah counting: Without large n*, Phys. Rev. D **85** (2012), no. 2, 026007.
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URL: `http://www.mpia.de/~mathar`

Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany