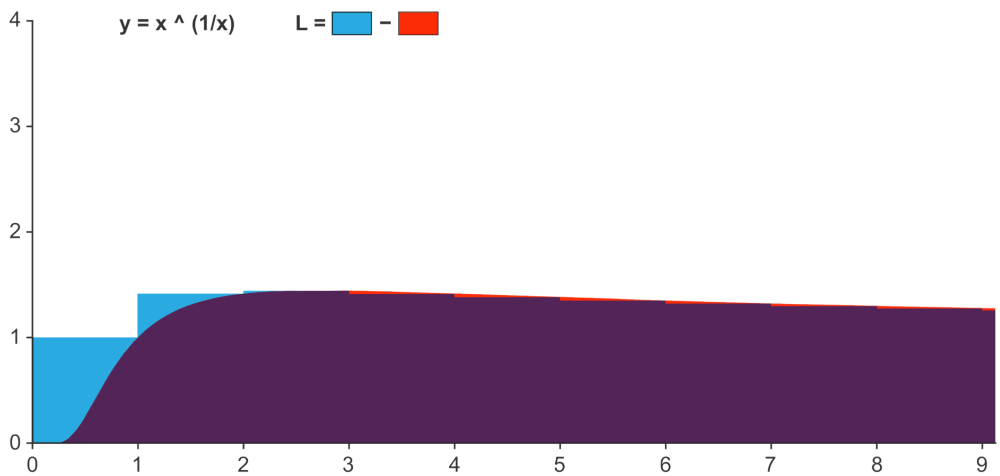
Accelerating a Limit.

Daniel Hoyt

$$L = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{\frac{1}{k}} - \int_0^n k^{\frac{1}{k}}\,dk\right)$$

L ~ 0.568180012359066....

It can be seen graphically:



Evaluating the limit directly is very computationally expensive. There are large mantissas and exponents, and the limit converges slowly. Here is an outline for a formula for this sum to converge faster than evaluating the limit directly.

We first translate the function down to be in the realm of convergence.

Consider this sum:

$$S = \sum_{k=2}^{\infty}(k^{1/k} - 1)$$

S still diverges because

$$k^{\frac{1}{k}} - 1 = e^{\frac{\ln k}{k}} - 1 > \frac{\log k}{k} > \frac{1}{k}$$

and $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k} = \infty$

Let's raise the second k to a power t so we can examine how tweaking that variable affects the output

$$S_t = \sum_{k=2}^{\infty}(k^{1/k^t} - 1)$$

.

Breaking t out in parts,

$$t = (1 + \frac{1}{x}),$$

$$S_x = \sum_{k=2}^{\infty}(k^{1/k^{1+1/x}} - 1).$$

The sum now converges for positive x, so we can play around a bit.

See that:

$$k^{1/k^t} = e^{(\log k)/k^t} = \sum_{n=0}^{\infty} \frac{(\log k)^n}{n!k^{nt}}$$

Let's define:

$$S_t = \sum_{k=2}^{\infty}\left(\left(\sum_{n=0}^{\infty} \frac{(\log k)^n}{n!k^{nt}}\right) - 1\right)$$

Notice when n=0 we get 1, so it cancels the -1 term resulting in:

$$\sum_{k=2}^{\infty}\sum_{n=1}^{\infty} \frac{(\log k)^n}{n!k^{nt}}$$

What is effectively adding column by column instead of row by row, we get:

$$\sum_{n=1}^{\infty}\sum_{k=2}^{\infty} \frac{(\log k)^n}{n!k^{nt}} = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=2}^{\infty} \frac{(\log k)^n}{k^{nt}}$$

Using the definition

$$\zeta^{(n)}(t) = e^{i\pi n}\sum_{k=2}^{\infty} \frac{\log k^n}{k^t}$$ valid for all $n \in \mathbb{R}, t \in \mathbb{C}$,

we get:

$$S_t = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\zeta^{(n)}(nt)$$

Since we'll be looking at the Zeta function close to it's pole at 1, let's expand ζ with the Laurent expansion of the Zeta function expanded around 1. γ represents the Stieltjes constants. Where

$$\gamma_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} - \int_1^m \frac{(\log x)^n}{x}dx\right)$$

Expansion of Zeta:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}(-1)^n \frac{\gamma_n}{n!}(s-1)^n$$

Giving us

$$\zeta^{'}(s) = -\frac{1}{(s-1)^2} - \gamma_1 + \gamma_2(s-1) - \frac{1}{2}\gamma_3(s-1)^2 \cdots$$

$$\zeta^{''}(s) = \frac{2}{(s-1)^3} + \gamma_2 - \gamma_3(s-1) + \frac{1}{2}\gamma_4(s-1)^2 \cdots$$

$$\zeta^{'''}(s) = -\frac{6}{(s-1)^4} - \gamma_3 + \gamma_4(s-1) - \frac{1}{2}\gamma_5(s-1)^2 ...$$

Using these interpretations and setting $s = n(1 + \frac{1}{x})$ and examining what occurs when $x \to \infty$ allows us to see why the output of S_x is always x^2 plus a constant less than 1. As before, we add column by column instead of row by row for the summations.

Examining the largest term when n=1, we have:

$$\frac{1}{((1+\frac{1}{x})-1)^2} = x^2$$

x^2 obviously diverges, so already we can tell the original sum S diverges and keep track of x^2. Let's keep going to see how the remainder term behaves as $x \to \infty$, $t = 1$. When n=2 and greater, the terms such as

$$\frac{2}{(s-1)^3} = \frac{2}{(n-1)^3}$$

turn into finite values. Summing all the terms in S_t, setting $t = 1 + \frac{1}{x}$ and letting x zoom off, we arrive at the limiting behavior of the sum S_x.

$$\lim_{x\to\infty}\left(\left(\sum_{k=2}^{\infty}(k^{1/k^{1+1/x}}-1)\right) - x^2\right) = \sum_{n=2}^{\infty}\frac{1}{(n-1)^{n+1}} + \sum_{n=1}^{\infty}\frac{\gamma_n}{n!} + \sum_{k=3}^{\infty}\sum_{n=k}^{\infty}(-1)^k\frac{\gamma_n(n-k+1)^{k-2}}{(n-k+2)!(k-2)!} = C,$$

and $\sum_{n=1}^{\infty}\frac{\gamma_n}{n!} = \frac{1}{2} - \gamma_0$, because $\zeta(0) = -\frac{1}{2} = -\frac{1}{1} + \sum_{n=0}^{\infty}\frac{\gamma_n}{n!}$

So,

$$\lim_{x\to\infty}\left(\left(\sum_{k=2}^{\infty}(k^{1/k^{1+1/x}}-1)\right)-x^2\right)=\sum_{n=2}^{\infty}\frac{1}{(n-1)^{n+1}}+\sum_{k=3}^{\infty}\sum_{n=k}^{\infty}(-1)^k\frac{\gamma_n(n-k+1)^{k-2}}{(n-k+2)!(k-2)!}+\frac{1}{2}-\gamma_0=C$$

γ_n is bounded by $|\gamma_n|<\frac{n!}{2^{n+1}}$, and numerically it stabilizes within the realm of testing.

$$C=0.98854960114226875064\ldots$$

So it seems

$$\sum_{k=2}^{\infty}(k^{1/(k^{1+1/x})}-1)=x^2+C_x,$$

where $C_1\le C_x\le C$, $Re(x)\ge 1$,
$C_1=-0.028501\ldots$. C_x is positive when $x>1.38$.

Since C_x is positive when $x>1.38$, it appears

$$\sum_{k=2}^{\infty}(k^{1/(k^{1+1/x})}-1)=\lfloor x^2\rfloor$$

is true when x is the square root of any natural number greater than 1.

Here are some examples:
$S_4=16.238932773$,
$S_{12}=144.5937831$,
$S_{\sqrt{1729}}=1729.84841$,
$S_{50000}=2500000000.988421705$.

Also note that:

$$\int_1^{\infty}\left(k^{1/k^t}-1\right)dk=\sum_{n=1}^{\infty}\frac{1}{n!}\int_1^{\infty}(\log k)^n\,k^{-nt}dk=$$

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}y^n e^{(1-nt)y}dy=\sum_{n=1}^{\infty}\frac{1}{(nt-1)^{n+1}}.$$

If $t=1+1/x$ then the main contribution to this sum is the term n = 1 which gives a contribution of x^2 and so the sum is equal to

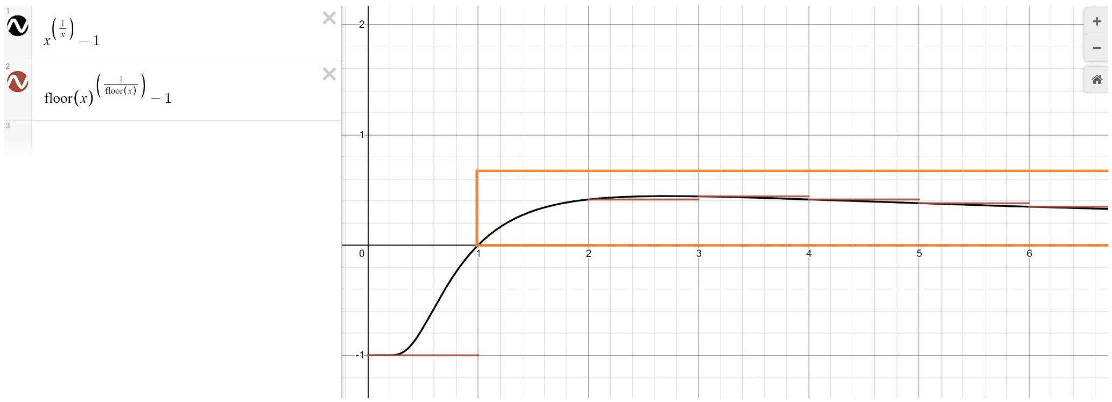
$$x^2+\sum_{n=2}^{\infty}\frac{1}{(nt-1)^{n+1}}.$$

Looking at the difference as $x \to \infty$,

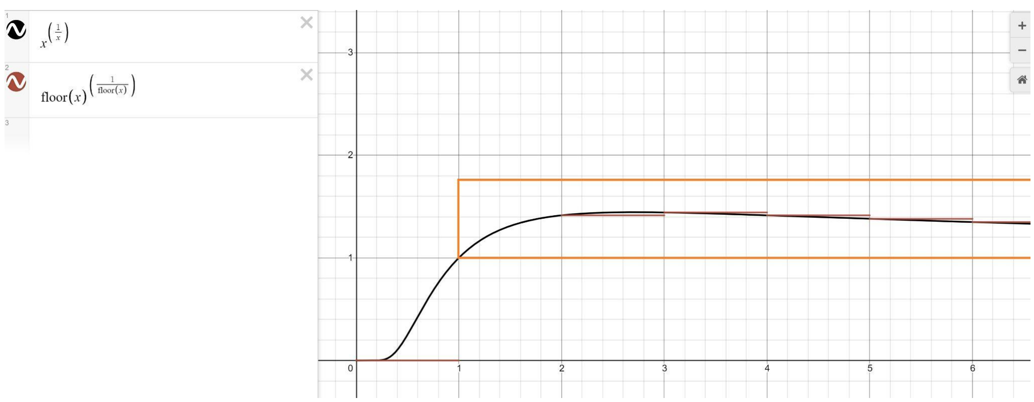
$$\lim_{x\to\infty}\left(\int_1^{\infty}\left(k^{1/k^{1+\frac{1}{x}}}-1\right)dk-\sum_{k=1}^{\infty}(k^{1/k^{1+\frac{1}{x}}}-1)\right)=\gamma_0-\frac{1}{2}-\sum_{k=3}^{\infty}\sum_{n=k}^{\infty}(-1)^k\frac{\gamma_n(n-k+1)^{k-2}}{(n-k+2)!(k-2)!}.$$

As $x \to \infty$, $k^{\frac{1}{k^{1+\frac{1}{x}}}} = k^{\frac{1}{k}}$

Looking at these two functions:



A translation up 1 has no effect of the limiting difference:



So we can say:

$$\lim_{n\to\infty}\left(\int_1^{n+1}k^{\frac{1}{k}}\,dk-\sum_{k=1}^{n}k^{\frac{1}{k}}\right)=\gamma_0-\frac{1}{2}-\sum_{k=3}^{\infty}\sum_{n=k}^{\infty}(-1)^k\frac{\gamma_n(n-k+1)^{k-2}}{(n-k+2)!(k-2)!}$$

And

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}k^{\frac{1}{k}}-\int_1^{n}k^{\frac{1}{k}}\,dk\right)=\frac{3}{2}-\gamma_0+\sum_{k=3}^{\infty}\sum_{n=k}^{\infty}(-1)^k\frac{\gamma_n(n-k+1)^{k-2}}{(n-k+2)!(k-2)!}$$

So,

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{\frac{1}{k}} - \int_0^n k^{\frac{1}{k}}dk\right) = \frac{3}{2} - \gamma_0 - \int_0^1 k^{\frac{1}{k}}dk + \sum_{k=3}^{\infty}\sum_{n=k}^{\infty}(-1)^k \frac{\gamma_n (n-k+1)^{k-2}}{(n-k+2)!(k-2)!}$$

Now we are prepared to compute this limit.

For comparison, evaluating the limit directly with Wolfram Alpha with large n, close to where if n was larger the site would error out, gets us only to 3 decimal places.
For n = 60000, we get an approximation of ~0.568… (Not enough for an OEIS entry!). Even with Pari GP with n=100000000 (\p200), we only get ~0.568180…

Let’s use the power of our new formula with Python to compute the limit.

Using Python 3 we can get 15 significant figures in a few seconds with a few lines of code:

```
—-------------
from mpmath import stieltjes,fac,quad

def limgen(n):
    terms = []
    for y in range(3, n):
        for x in range(y, n):
            terms.append((((-1)**y)*stieltjes(x)*(x-(y-1))**(y-2))/(fac(x-(y-2))*fac(y-2)))
    return terms

f = lambda x: x**(1/x)
int01 = quad(f, [0,1])
limit = sum(limgen(60)) + 1.5 - stieltjes(0) - int01
print(limit)
—---------------
```

This small program gives an approximation of 0.568180012359066… and more can be computed with precision libraries easily. Python uses complex integration to get the Stieltjes constants, which is very fast up to a few thousand decimal places. The Python program is still a simple implementation - if we were serious about computing this limit to millions of figures or more we would need to implement binary splitting algorithms and parallelization for the Stieltjes constants and main computation. For now this program can give us hundreds of digits without too much modification.