Lemma. (1) The only solutions to

$$x^4 - 2y^2 = 1$$

are $(x, y) = (\pm 1, 0)$.
(2) The only solutions to

$$x^4 - 2y^2 = -1$$

are $(x, y) = (\pm 1, \pm 1)$.
(3) The only solutions to

$$x^2 - 2y^4 = 1$$

are $(x, y) = (\pm 1, 0)$.
(4) The only solutions to

$$x^2 - 2y^4 = -1$$

are $(x, y) = (\pm 1, \pm 1), (\pm 239, \pm 13)$.

Proof. (1) Let $X = 2x^2$, $Y = 4xy$, then

$$Y^2 = X^3 - 4X.$$

Type `E=ellinit([0,0,0,-4,0]); ellanalyticrank(E); elltors(E);` in PARI/GP to see that this is an elliptic curve with rank 0 and torsion group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so the only solutions (X, Y) are $(X, Y) = (0, 0), (\pm 2, 0)$, corresponding to $(x, y) = (\pm 1, 0)$. (See also `https://www.lmfdb.org/EllipticCurve/Q/64/a/3` for the integral points on the elliptic curve).

Alternative elementary solution. Let

$$L_0 = 1, \quad L_1 = 1, \quad L_{n+2} = 2L_{n+1} + L_n, \quad \forall n \in \mathbb{N} \quad (\text{A001333});$$
$$P_0 = 0, \quad P_1 = 1, \quad P_{n+2} = 2P_{n+1} + P_n, \quad \forall n \in \mathbb{N} \quad (\text{A000129}),$$

then all solutions to $x^2 - 2y^2 = 1$ are given by $(x, y) = (L_{2k}, P_{2k})$, and all solutions to $x^2 - 2y^2 = -1$ are given by $(x, y) = (L_{2k+1}, P_{2k+1})$. Note that $L_{2k} = 2L_k^2 + (-1)^{k-1}$, so if $L_{2k} = x_0^2$ is a square, then (x_0, L_k) itself satisfies $x^2 - 2y^2 = (-1)^{k-1}$, so $(x_0, L_k) = (L_m, P_m)$ for some $m \equiv k - 1 \pmod 2$. But we have $2 = P_2 < L_2 < P_3 < L_3 < \cdots$, so the only possibility is $k = 0$ and $m = 1$, and $x_0 = 1$.

(2) Let $X = 2x^2$, $Y = 4xy$, then

$$Y^2 = X^3 + 4X.$$

Type `E=ellinit([0,0,0,4,0]); ellanalyticrank(E); elltors(E);` in PARI/GP to see that this is an elliptic curve with rank 0 and torsion group $\mathbb{Z}/4\mathbb{Z}$, so the only solutions (X, Y) are $(X, Y) = (0, 0), (2, \pm 4)$, corresponding to $(x, y) = (\pm 1, \pm 2)$. (See also `https://www.lmfdb.org/EllipticCurve/Q/32/a/4` for the integral points on the elliptic curve).

(3) Let $X = 2y^2$, $Y = 2xy$, then

$$Y^2 = X^3 + 2X.$$

Type `E=ellinit([0,0,0,2,0]); ellanalyticrank(E); elltors(E);` in PARI/GP to see that this is an elliptic curve with rank 0 and torsion group $\mathbb{Z}/2\mathbb{Z}$, so the only solutions (X,Y) are $(X,Y)=(0,0)$, corresponding to $(x,y)=(\pm 1,0)$. (See also https://www.lmfdb.org/EllipticCurve/Q/256/c/2 for the integral points on the elliptic curve).

Alternative elementary solution. Note that $\gcd(L_n,P_n)=1$ for all $n\in\mathbb{N}$ (where $\{L_n\}$ and $\{P_n\}$ are defined in the proof of (1)), and $P_{2k}=2L_kP_k$. If P_{2k} is a square, then since L_k is odd, both L_k and $2P_k$ must be squares. We deduce then that k is even (otherwise $2P_k\equiv 2 \pmod 4$). But (1) shows that L_k cannot be a square for even k unless $k=0$.

(4) Let $X=2y^2$, $Y=2xy$, then

$$Y^2=X^3-2X.$$

This is an elliptic curve with rank 1, and the integral points are given in https://www.lmfdb.org/EllipticCurve/Q/256/b/1 as $(X,Y)=(-1,\pm 1),(0,0),(2,\pm 2),(338,\pm 6214)$, corresponding to $(x,y)=(\pm 1,\pm 1),(\pm 239,\pm 13)$. (See https://www.impan.pl/en/publishing-house/journals-and-series/colloquium-mathematicum/all/109/1/87140 for a complete proof of the integral points on the elliptic curve). □

Theorem. Let N be a even number such that $N/2$ has only prime factors congruent to 1 modulo 4, then

$$d(p^2-1)=N$$

has no odd solutions p other than $d(7^2-1)=10$ and $d(9^2-1)=10$, where $d(n)$ (A000005) is the number of divisors of n.

Proof. In general, let

$$N_{m,r}:=\{N\ :\ \text{all except one prime factors of } N \text{ are congruent to } 1 \text{ modulo } m,\\ \text{and one prime factor of } N \text{ is congruent to } 1+r \text{ modulo } m\},\quad 0\le r\le m-1,$$

then $d(n)\in N_{m,r}$ implies that $n=m^kp^r$ for some $m\in\mathbb{N}$ and prime p. Indeed, if we write $N\in N_{m,r}$ as a product of numbers ≥ 2 arbitrarily, then one factor is congruent to $1+r$ modulo m and the others are congruent to 1 modulo m, so $d(n)=N$ implies that one prime factor of n has multiplicity congruent to r modulo m and the other prime factors have multiplicities divisible by m.

Here we have $N\in N_{4,1}$. Since $8\mid p^2-1$ for odd p, if $d(p^2-1)=N$, then

$$p^2-1=16m^4P$$

for $m\in\mathbb{N}$ and prime P. (Here m can be even, and P can be a factor of $2m$). Since $\gcd(p-1,p+1)=2$, we have

$$\begin{cases} p\pm 1=8M^4P \\ p\mp 1=2N^4 \end{cases} \quad\text{or}\quad \begin{cases} p\pm 1=8M^4 \\ p\mp 1=2N^4P \end{cases}$$

for some $\gcd(2MP,N)=1$ in the first case and $\gcd(2M,NP)=1$ in the second case. In the first case we have

$$N^4-4M^4P=\pm 1,$$

so in particular $N^4 - 4M^2P = 1$ by modulo 4, or $(N^2-1)(N^2+1) = 4M^4P$. Since $\gcd(N^2-1, N^2+1) = 2$, we have

$$\begin{cases} N^2 \pm 1 = 2M_1^4 P \\ N^2 \mp 1 = 2M_2^4 \end{cases}$$

which has only solutions $(M_2, N) = (13, 239)$ by the parts (3) (4) in the Lemma since $N > 1$, corresponding to $M_1^4 P = 28560$ which is impossible. In the second case we have

$$N^4P - 4M^4 = \pm 1.$$

If $N^4P = 4M^4 - 1 = (2M^2-1)(2M^2+1)$, then $\gcd(2M^2-1, 2M^2+1) = 1$ implies that

$$\begin{cases} 2M^2 \pm 1 = N_1^4 P \\ 2M^2 \mp 1 = N_2^4 \end{cases}$$

which has only solutions $(M, N_2) = (1, 1)$ by the parts (1) (2) in the Lemma, corresponding to $M_1 = 1$ and $P = 3$. If $N^4P = 4M^4 + 1 = (2M^2 - 2M + 1)(2M^2 + 2M + 1)$, then $\gcd(2M^2-2M+1, 2M^2+2M+1) = \gcd(2M^2-2M+1, 4M) = \gcd(2M^2-2M+1, M) = 1$ implies that

$$\begin{cases} 2M^2 \mp 2M + 1 = N_1^4 P \\ 2M^2 \pm 2M + 1 = N_2^4 \end{cases}$$

or

$$(2M \pm 1)^2 - 2N_2^4 = -1.$$

The solutions to the last equation are $(M, N_2) = (1, 1), (119, 13), (120, 13)$ by the parts (4) in the Lemma, corresponding to $N_1^4 P = 5, 28085, 29041$ where the last two are not possible. In conclusion, we have

$$\begin{cases} p + 1 = 8 \\ p - 1 = 6 \end{cases} \quad \text{or} \quad \begin{cases} p - 1 = 8; \\ p + 1 = 10, \end{cases}$$

so $p = 7$ or $p = 9$. □

Similarly, let $r \geq 3$ be an odd number, and suppose that N be a even number such that $N/2$ has only prime factors congruent to 1 modulo $2r$, then if $d(p^2-1) = N$ for some odd p, then

$$p^2 - 1 = 2^{2r} m^{2r} P$$

for $m \in \mathbb{N}$ and prime P, and we have

$$\begin{cases} p \pm 1 = 2^{2r-1} M^{2r} P \\ p \mp 1 = 2N^{2r} \end{cases} \quad \text{or} \quad \begin{cases} p \pm 1 = 2^{2r-1} M^{2r}; \\ p \mp 1 = 2N^{2r} P, \end{cases}$$

corresponding to

$$\begin{aligned} p = 2N^{2r} + 1, &\quad 2^{2r-2} M^{2r} P = N^{2r} - 1; \\ p = 2^{2r-1} M^{2r} - 1, &\quad N^{2r} P = 2^{2r-2} M^{2r} - 1; \\ p = 2^{2r-1} M^{2r} + 1, &\quad N^{2r} P = 2^{2r-2} M^{2r} + 1 \end{aligned}$$

(note that $N^{2r} - 2^{2r-2} M^{2r} P = -1$ is impossible modulo 2^{2r-2}). We have then

$$p = 2N^{2r} + 1, \quad \begin{cases} 2^{2r-3} M_1^{2r} P &= N^r - 1; \\ 2M_2^{2r} &= N^r + 1, \end{cases} \tag{1}$$

$$p = 2N^{2r} + 1, \quad \begin{cases} 2^{2r-3} M_1^{2r} &= N^r - 1; \\ 2M_2^{2r} P &= N^r + 1, \end{cases} \tag{2}$$

$$p = 2^{2r-1} M^{2r} - 1, \quad \begin{cases} N_1^{2r} P &= 2^{r-1} M^r - 1; \\ N_2^{2r} &= 2^{r-1} M^r + 1, \end{cases} \tag{3}$$

$$p = 2^{2r-1} M^{2r} + 1, \quad N^{2r} P = 2^{2r-2} M^{2r} + 1 \tag{4}$$

(note that $\begin{cases} 2^{2r-3} M_1^{2r} P &= N^r + 1 \\ 2M_2^{2r} &= N^r - 1 \end{cases}$ and $\begin{cases} 2^{2r-3} M_1^{2r} &= N^r + 1 \\ 2M_2^{2r} P &= N^r - 1 \end{cases}$ is impossible modulo 2^{2r-4} in (**??**) and (**??**), and $\begin{cases} N_1^{2r} P &= 2^{r-1} M^r + 1 \\ N_2^{2r} &= 2^{r-1} M^r - 1 \end{cases}$ is impossible modulo 2^{r-1} in (**??**)). But:

- $2M_2^{2r} = N^r + 1$ in (**??**) can be written as $x^r - 2y^r = -1$, which is impossible by "Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$" https://personal.math.ubc.ca/~bennett/B-Crelle2.pdf;
- $2^{2r-3} M_1^{2r} = N^r - 1$ in (**??**) can be written as $x^r - 2y^2 = 1$ for $y = 2^{r-2} M_1^r$, which is impossible by "A Note on Two Diophantine Equations" https://www.m-hikari.com/ams/ams-2012/ams-133-136-2012/taoAMS133-136-2012-2.pdf;
- From $N_2^{2r} = 2^{r-1} M^r + 1$ in (**??**), we get

  $$\begin{cases} N_2^r \pm 1 &= 2^{r-2} M^r; \\ N_2^r \mp 1 &= 2M^r, \end{cases}$$

  and the second line is impossible by "Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$".

In conclusion: Only (**??**) is nontrivial, and p must be of the form $p = 2^{2r-1} M^{2r} + 1$.