**Question.** Let

$$S := \{N \text{ even} : \Omega(N) \le 3\} \cup \{16, 24, 36, 54\},$$

where $\Omega(n)$ (A001222) is the number of prime factors of $n$ counted with multiplicity. For each $N \in S$, find all $p \equiv 1, 5 \pmod 6$ such that $d(p^2 - 1) = N$, where $d(n)$ (A000005) is the number of divisors of $n$.

**Lemma 1.** The only solutions to

$$3^m - 2^n = \pm 1$$

are $(m, n) = (0, 1), (1, 1), (1, 2), (2, 3)$.

*Proof.* Suppose that $n \ge 3$. Since

$$3^m \equiv 1, 3 \pmod 8,$$

we must have $3^m - 2^n = 1$. But $3^m \equiv 1 \pmod{2^n}$ implies that

$$2^{n-2} \mid m,$$

so $3^t - 4t \le 1$ for $t = 2^{n-2} \ge 2$, and then we get $(m, n) = (2, 3)$. □

**Lemma 2.** The only solution to

$$3^m - 2^n P = \pm 1, \quad n \ge 3, \quad P \text{ prime}$$

is $(m, n, P) = (4, 4, 5)$.

*Proof.* Similarly as in Lemma 1 we have $3^m - 2^n P = 1$, $2^{n-2} \mid m$, and

$$\frac{3^{2^{n-2}} - 1}{2^n} \,\Bigg|\, \frac{3^m - 1}{2^n} = P.$$

Let $T_n := \dfrac{3^{2^{n-2}} - 1}{2^n}$ (A068531) for $n \ge 3$, then $T_n \mid T_{n+1}$, so $T_n$ is not prime for $n \ge 5$. If $n = 4$, then $(m, P) = (4, 5)$. If $n = 3$, write $m = 2k$, then

$$P = \frac{3^{2k} - 1}{8} = \frac{(3^k - 1)(3^k + 1)}{8},$$

impossible. □

**Lemma 3.** The only solutions to

$$3^m P - 2^n = \pm 1, \quad m \ge 2, \quad P \text{ prime}$$

is $(m, n, P) = (2, 6, 7), (3, 9, 19)$.

*Proof.* We have $2^n \equiv \pm 1 \pmod{3^m}$, so

$$3^{m-1} \mid n,$$

and

$$\frac{2^{3^{m-1}}+1}{3^m} \,\Bigg|\, \frac{2^n-(-1)^n}{3^m} = P.$$

Let $T_m := \dfrac{2^{3^{m-1}}+1}{3^m}$ (A234039) for $m \geq 1$, then $T_m \mid T_{m+1}$, so $T_m$ is not prime for $m \geq 4$. If $m = 3$, then $(n, P) = (9, 19)$. If $m = 2$, write $n = 3k$, then

$$P = \frac{2^{3k}-(-1)^k}{9} = \frac{(2^k-(-1)^k)(2^{2k}+(-2)^k+1)}{9},$$

and the only possiblility is $(n, P) = (6, 7)$. □

**Lemma 4.** The only solutions to

$$3 \times 2^n \pm 1 = m^k, \quad k \geq 2$$

are $(n, m, k) = (0, 2, 2), (3, 5, 2), (4, 7, 2)$.

*Proof.* Suppose that $n \geq 3$. We have $m^k \equiv \pm 1 \pmod{2^n}$. If $k$ is odd, then

$$m \equiv (\pm 1)^{k^{-1} \bmod 2^{n-2}} = \pm 1 (\text{mod } 2^n),$$

so $m \geq 2^n - 1$, and $3t + 1 \geq (t-1)^3$ for $t = 2^n \geq 8$, which is impossible. So $k$ is even, and we can suppose that $k = 2$. Then the equation implies that

$$3 \times 2^n + 1 = m^2, \quad m \equiv \pm 1 \pmod{2^{n-1}},$$

so $m \geq 2^{n-1} - 1$, and $6t + 1 \geq (t-1)^2$ for $t = 2^{n-1} \geq 4$, and then we get $(n, m, k) = (3, 5, 2), (4, 7, 2)$. □

**Lemma 5.** The only solutions to

$$3x^2 - 2^n = \pm 1$$

are $(n, x) = (0, 0), (1, 1), (2, 1)$.

*Proof.* If $n \geq 3$, then $3x^2 \equiv 3 \pmod 8$, impossible. □

Note that $24 \mid (p^2 - 1)$, so if $d(p^2 - 1) = N$ with $N \in S$, then $p^2 - 1$ has at most three distinct prime factors, since

$$S = \mathbb{N}^* \setminus \{e_1 \cdots e_\ell \,:\, \ell \geq 4, e_i \geq 2\,(i = 2, \cdots, \ell), e_1 \geq 4\}.$$

*Case 1.* $p^2 - 1$ has only two distinct factors. Then $p^2 - 1 = 2^i \times 3^j$ for $i \geq 3$, $j \geq 1$. Since $\gcd(p-1, p+1) = 2$, we have

$$\begin{cases} p \mp 1 = 2^{i-1}; \\ p \pm 1 = 2 \times 3^j, \end{cases}$$

so $3^j - 2^{i-2} = \pm 1$. This implies

$$p^2 - 1 = 2^3 \times 3^1, 2^4 \times 3^1, 2^5 \times 3^2 = 24, 48, 288,$$

by Lemma 1, corresponding to

$$(N, p) = \underline{(8, 5), (10, 7), (18, 17)}.$$

*Case 2.* $p^2 - 1$ has three distinct factors. If $\Omega(N) = 3$, there must be a factor with multiplicity 1 since $N$ has the factor 2. Write $p^2 - 1 = 2^i \times 3^j \times P$ ($j \geq 1$) or $2^i \times 3 \times P^j$ ($j \geq 2$) for $i \geq 3$ and a prime $P \geq 5$. Note that the three prime factors of $N$ are then $i + 1$, $j + 1$, and 2, so $i \geq 4$ is even, and $j$ is even if $j \geq 2$. If $N = 16, 24, 36$, or $54$, then

$$\begin{aligned} p^2 - 1 = {} & 2^3 \times 3 \times P, 2^5 \times 3 \times P, 2^3 \times 3^2 \times P, 2^3 \times 3 \times P^2, \\ & 2^8 \times 3 \times P, 2^5 \times 3^2 \times P, 2^5 \times 3 \times P^2, 2^3 \times 3^2 \times P^2, \\ & 2^8 \times 3^2 \times P, 2^8 \times 3 \times P^2, 2^5 \times 3^2 \times P^2, \end{aligned}$$

so other than

$$p^2 - 1 = 2^3 \times 3^2 \times P^2, 2^5 \times 3^2 \times P^2,$$

$p^2 - 1$ is also of the form $2^i \times 3^j \times P$ ($j \geq 1$) or $2^i \times 3 \times P^j$ ($j \geq 2$), where $(i, j) = (3, 1), (3, 2), (5, 1), (5, 2), (8, 1), (8, 2)$.

We have

$$\begin{cases} p \pm 1 = 2^{i-1} \\ p \mp 1 = 2 \times 3^j \times P \end{cases} \qquad \begin{cases} p \pm 1 = 2^{i-1} \times 3^j \\ p \mp 1 = 2 \times P \end{cases} \qquad \begin{cases} p \pm 1 = 2^{i-1} \times P \\ p \mp 1 = 2 \times 3^j \end{cases}$$

$$\begin{cases} p \pm 1 = 2^{i-1} \\ p \mp 1 = 2 \times 3 \times P^j \end{cases} \qquad \begin{cases} p \pm 1 = 2^{i-1} \times 3; \\ p \mp 1 = 2 \times P^j, \end{cases}$$

corresponding to

$$3^j \times P - 2^{i-2} = \pm 1; \tag{1}$$
$$P - 2^{i-2} \times 3^j = \pm 1; \tag{2}$$
$$3^j - 2^{i-2} \times P = \pm 1; \tag{3}$$
$$3 \times P^j - 2^{i-2} = \pm 1 \, (j \geq 2); \tag{4}$$
$$P^j - 2^{i-2} \times 3 = \pm 1 \, (j \geq 2), \tag{5}$$

Note that (4) is impossible by Lemma 5 since $j$ is even, and (5) implies

$$p^2 - 1 = 2^5 \times 3 \times 5^2, 2^6 \times 3 \times 7^2 = 2400, 9408,$$

by Lemma 4, corresponding to

$$(N, p) = \underline{(36, 49), (42, 97)}.$$

For (3), note that $(i,j)=(3,2)$ gives the solution

$$p^2-1=2^3\times 3^2\times 5=360,$$

corresponding to $(N,p)=\underline{(24,19)}$. If $i\geq 5$, then Lemma 2 tells that

$$p^2-1=2^6\times 3^4\times 5=25920,$$

corresponding to $(N,p)=\underline{(70,161)}$. If $i=4$, then

$$P=\frac{3^j-(-1)^j}{4}.$$

But $U_j:=\dfrac{3^j-(-1)^j}{4}$ (A015518) forms a divisibility sequence ($U_j\mid U_k$ if and only if $j\mid k$), so $j$ must itself be prime, which implies $j=2$ and $P=2$, contradicting $P\geq 5$.

For (1), if $j\geq 2$, then Lemma 3 tells that

$$p^2-1=2^8\times 3^2\times 7, 2^{11}\times 3^3\times 19=16128, 1050624,$$

corresponding to $(N,p)=\underline{(54,127)}, (96,1025)$. If $j=1$, and

$$P=\frac{2^{i-2}-(-1)^{i-2}}{3}.$$

But $U_i:=\dfrac{2^i-(-1)^i}{3}$ (A001045) forms a divisibility sequence, so either $i-2=4$, either $i-2\geq 3$ must itself be prime. Since $i$ is either $3,5$ or even, we see that $i=6$ ($i=5$ gives $P=3$), so

$$p^2-1=2^6\times 3^1\times 5+1=960,$$

corresponding to

$$(N,p)=\underline{(28,31)}.$$

At last, we see that (2) is the only nontrivial equation, and we consider separately

$$P=2^{i-2}\times 3^j-1,\quad p=2^{i-1}\times 3^j-1; \tag{2'}$$
$$P=2^{i-2}\times 3^j+1,\quad p=2^{i-1}\times 3^j+1. \tag{2''}$$

If $N=16,24,36,$ or $54$, meaning that $(i,j)=(3,1),(5,1),(3,2),(5,2),(8,1),(8,2)$, then

$$\begin{aligned}(P,p)=&(5,11),(7,13),(17,35),(19,37),(23,47),\\&(71,143),(73,145),(191,383),(193,385),(577,1153),\end{aligned}$$

corresponding to

$$\begin{aligned}(N,p)=&\underline{(16,11),(16,13),(24,35),(24,37),(24,47)},\\&\underline{(36,143),(36,145),(36,383),(36,385),(54,1153)}.\end{aligned}$$

For $\Omega(N)=3$, note that (2′) can only have solution when $j=1$, otherwise $2^{i-2}\times 3^j$ is a square. Doing the very same process to

$$p^2-1=2^3\times 3^2\times P^2, 2^5\times 3^2\times P^2,$$

we can see that there are no solutions corresponding to these two cases.

**Conclusion.** Let $K(N)$ be the set of $p \equiv 1, 5 \pmod 6$ such that $d(p^2-1) = N$ for

$$K \in S = \{N \,\text{even} \,:\, \Omega(N) \le 3\} \cup \{16, 24, 36, 54\}.$$

Then:

$$\begin{aligned}
K(10) = \{7^\star\}, \quad K(N) = \emptyset \,\text{for}\, \Omega(N) \le 2, N \ne 10;\\
K(8) = \{5^\star\}, \quad K(12) = \emptyset, \quad K(18) = \{17^\star\},\\
K(28) = \{31^\star, 95\}, \quad K(42) = \{97^\star\}, \quad K(70) = \{161, 2593^\star, 5833\};\\
K(16) = \{11^\star, 13^\star\}, \quad K(24) = \{19^\star, 35, 37^\star, 47^\star\},\\
K(36) = \{49, 143, 145, 383^\star, 385\}, \quad K(54) = \{127^\star, 1153^\star\}
\end{aligned}$$

($\star$ corresponds to primes). For other $N$ with $\Omega(N) = 3$, write $N = 2qr$ with $q \le r$ primes, then each solution corresponds to one of the three combinations $(P, p)$ with prime value $P$:

$$\begin{aligned}
P &= 2^{r-3} \times 3 - 1, & p &= 2^{r-2} \times 3 - 1 \,(q = 2);\\
P &= 2^{r-3} \times 3^{q-1} + 1, & p &= 2^{r-2} \times 3^{q-1} + 1;\\
P &= 2^{q-3} \times 3^{r-1} + 1, & p &= 2^{q-2} \times 3^{r-1} + 1 \,(r > q \ge 5).
\end{aligned}$$

(This is what gives the additional $(N, p) = (28, 95), (70, 2593), (70, 5833)$). In particular $|K(N)| \le 2$ for each certain $N$.

**Conjecture 1.** Suppose that $\Omega(N) = 3$. Other than

$$\begin{aligned}
K(20) = \{23^\star, 25\}, \quad K(28) = \{31^\star, 95\}, \quad K(70) = \{161, 2593^\star, 5833\},\\
K(182) = \{1492993^\star, 17006113\},
\end{aligned}$$

we have $|K(N)| \le 1$; in other words, we have

$$r \,\text{prime}, \quad 2^{r-3} \times 3 - 1, 2^{r-3} \times 3 + 1 \,\text{both primes} \Longrightarrow r = 5$$

and

$$r > q \ge 5 \,\text{primes}, \quad 2^{r-3} \times 3^{q-1} + 1, 2^{q-3} \times 3^{r-1} + 1 \,\text{both primes} \Longrightarrow (q, r) = (5, 7), (7, 13).$$

Note that if we require $p$ to be **prime** (not only $p \equiv 1, 5 \pmod 6$), then such solutions are very rare, because it does not happen very often that

$$2^i \times 3^j - 1, \quad 2^{i+1} \times 3^j - 1$$

or

$$2^i \times 3^j + 1, \quad 2^{i+1} \times 3^j + 1$$

turn out to be both primes. In fact, I conjecture that

**Conjecture 2.** The largest $N$ with $\Omega(N) = 3$ such that there exists some prime $p$ satisfying $d(p^2 - 1) = N$ is $N = 518$; in other words, we have

$$r\,\text{prime}, \quad 2^{r-3} \times 3 - 1, 2^{r-2} \times 3 - 1\,\text{both primes} \Longrightarrow r = 5,$$

$$r \geq q\,\text{primes}, \quad 2^{r-3} \times 3^{q-1} + 1, 2^{r-2} \times 3^{q-1} + 1\,\text{both primes} \Longrightarrow (q, r) = (3, 5), (5, 7), (7, 13),$$

and

$$r > q \geq 5\,\text{primes}, \quad 2^{q-3} \times 3^{r-1} + 1, 2^{q-2} \times 3^{r-1} + 1\,\text{both primes} \Longrightarrow (q, r) = (7, 37).$$

In particular, the complete list of solutions to

$$\Omega(N) = 3, \quad p\,\text{prime}, \quad d(p^2 - 1) = N$$

is

$$\begin{aligned}(N, p) = &(8, 5), (18, 17), (20, 23), (28, 31), (30, 73), (42, 97),\\ &(70, 2593), (182, 1492993), (518, 4803028329503971873).\end{aligned}$$

As an end, it is natural to guess that

**Conjecture 3.** For even $N$ with $\Omega(N) \geq 4$, if $N \neq 16, 24, 36,$ or $54$, then there exists infinitely many primes $p$ such that

$$d(p^2 - 1) = N.$$

Let's see what is needed in the conjecture for (perhaps the easiest) case $4 \mid N$, $\Omega(N) \geq 4$, if $N \neq 16, 24, 36$. We can write $N = (i+1)(j+1) \times 2 \times 2$ for $i \geq 3$ and $j \geq 1$, so it suffices to show that for every $i \geq 3$ and $j \geq 1$, there exists infinitely many triples or primes $(p, P, Q)$ such that

$$p^2 - 1 = 2^i \times 3^j \times PQ, \quad P, Q \geq 5, \quad P \neq Q;$$

in other words, such that

$$\begin{cases} p \pm 1 = 2^{i-1} \\ p \mp 1 = 2 \times 3^j \times PQ \end{cases} \quad \begin{cases} p \pm 1 = 2^{i-1} \times 3^j \\ p \mp 1 = 2 \times PQ \end{cases} \quad \begin{cases} p \pm 1 = 2^{i-1} \times PQ \\ p \mp 1 = 2 \times 3^j \end{cases}$$

$$\begin{cases} p \pm 1 = 2^{i-1} \times P \\ p \mp 1 = 2 \times 3^j \times Q \end{cases} \quad \begin{cases} p \pm 1 = 2^{i-1} \times 3^j \times P; \\ p \mp 1 = 2 \times Q, \end{cases}$$

corresponding to

$$i \neq 8, \quad j = 1, \quad PQ = \frac{2^{i-2} - (-1)^{i-2}}{3}, \quad p = 2^{i-1} + (-1)^{i-1};^{1} \tag{6}$$

$$PQ = 2^{i-2} \times 3^j \pm 1, \quad p = 2^{i-1} \times 3^j \pm 1; \tag{7}$$

$$i = 4, \quad PQ = \frac{3^j - (-1)^j}{4}, \quad p = 2 \times 3^j - (-1)^j; \tag{8}$$

$$3^j Q = 2^{i-2} P \pm 1, \quad p = 2^{i-1} P \pm 1; \tag{9}$$

$$Q = 2^{i-2} \times 3^j P \pm 1, \quad p = 2^{i-1} \times 3^j P \pm 1. \tag{10}$$

But in general, there is no polynomial $f$ such that $f(p)$ is proved to be prime infinitely often for primes $p$, so it may be hard to prove that equations of type (9) or (10) has infinitely many solutions $(p, P, Q)$ that are triples or primes. The case $4 \nmid N$ (e.g. $N = 90, 162$) may be even harder since $p^2 - 1$ can have at most one prime factor with multiplicity 1. (We have $d(p^2 - 1) = 90$ for primes

$$p = 199, 8713, 449353, 2626633, 11577673, 53127433, \\ 59754313, 149091913, 177698953, 213252553, 230437513, \cdots,$$

and $d(p^2 - 1) = 162$ for primes

$$p = 1151, 139393, 9124993, 26266753, 174321793, 202246273, \cdots.$$

[1] Actually this one is highly improbable: if $i-1$ is odd, then $i-1 = k$ must be a prime to make $p = 2^k - 1$ a prime, so $k \,\Big|\, \frac{2^{k-1} - (-1)^{k-1}}{3} = \frac{2^{k-1} - 1}{3}$, and $\frac{2^{k-1} - 1}{3k} = \frac{(2^{\frac{k-1}{2}} + 1)(2^{\frac{k-1}{2}} - 1)}{3k}$ must be prime, impossible unless $k = 7, 11$ (but $2^{11} - 1$ is not prime). So $i - 1$ must be even, then we must have $i - 1 = 2^k$ to make $p = 2^{2^k} + 1$ prime, and $PQ = \frac{2^{2^k - 1} + 1}{3}$, which in turn implies that $2^k - 1$ is prime ($2^k - 1$ cannot be a perfect power, and if $m, n$ are coprime odd numbers, then $\frac{2^m + 1}{3}, \frac{2^n + 1}{3} \,\Big|\, \frac{2^{mn} + 1}{3}$, which implies $\frac{2^{mn} + 1}{3} = \frac{2^m + 1}{3} \times \frac{2^n + 1}{3} \times (\text{something else})$).