The Grand Sum Of $n \times n \times n$ matrix whose elements start from 1 and get higher, the more they're at the center

When dealing with a 3D matrix of elements, we sometimes need to calculate the elements in a way that the closer they are to center, the more they are respected; then we can use that for creating 3D art or equation.

So What is such matrix anyway?

The matrix is made of 3 dimensions of length n.

Let's assume n is 5; Then if it were 1D, it would look like

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

And if it were 2D, it would look like

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 3 & 4 & 3 & 2 \\ 3 & 4 & 5 & 4 & 3 \\ 2 & 3 & 4 & 3 & 2 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

The grand sum of the 1D matrix would be A002620 - OEIS , and the 2D matrix would be A317614 - OEIS, but for the 3D, there's no sequence registered at the OEIS.

So what does the 3D matric look like?

Since it cannot be brought up to a 2D paper, it may be hard to show the matrix.

If we divide the $i \times j \times k$ 3D matrix M to a 1D array A of $i \times j$ Matrices , with array length of k, we can then show what the 3D matrix of $M_{xyz} = x\_y\_z$ looks like: (Let's say $i = 2, j = 3, k = 3$)

$$\text{(z=1):} \quad \begin{bmatrix} 1\_1\_1 & 2\_1\_1 \\ 1\_2\_1 & 2\_2\_1 \\ 1\_3\_1 & 2\_3\_1 \end{bmatrix}$$

$$\text{(z=2):} \quad \begin{bmatrix} 1\_1\_2 & 2\_1\_2 \\ 1\_2\_2 & 2\_2\_2 \\ 1\_3\_2 & 2\_3\_2 \end{bmatrix}$$

$$\text{(z=3):} \quad \begin{bmatrix} 1\_1\_3 & 2\_1\_3 \\ 1\_2\_3 & 2\_2\_3 \\ 1\_3\_3 & 2\_3\_3 \end{bmatrix}$$

We can see that for the third dimension (z), I wrote a new 2D matrix for the layer of the matrix where z is for example 1, or 2 etc.

The *3D* Matrix whose elements start from 1 and get higher, the more they're at the center:

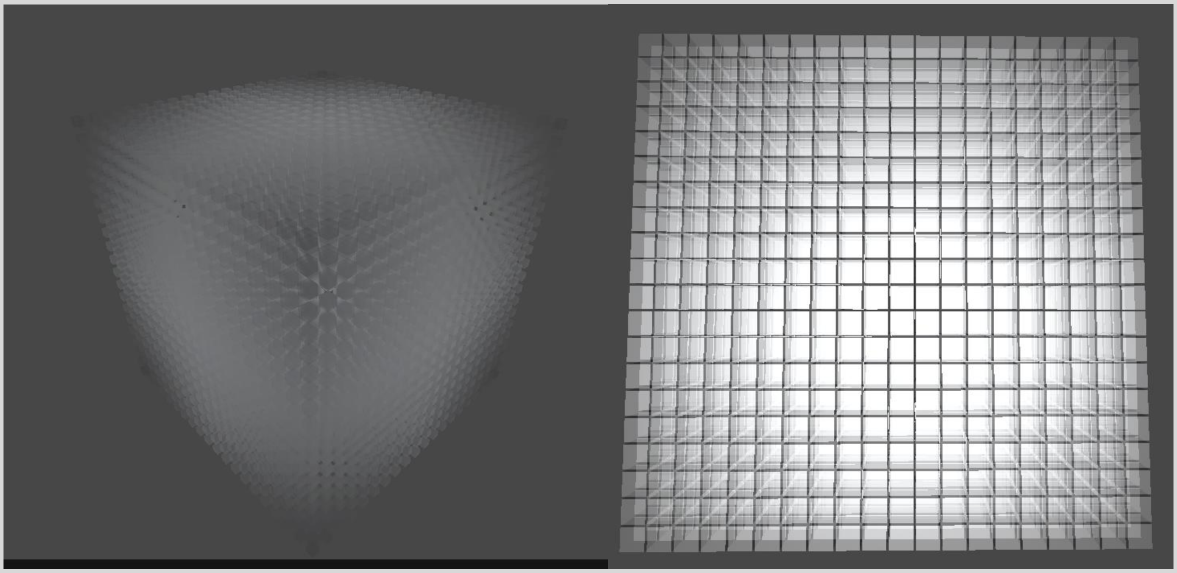
It would simply look like this, given $n = 3$:

$$\text{z = 1:} \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\text{z = 2:} \quad \begin{bmatrix} 2 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 2 \end{bmatrix}$$

$$\text{z = 3:} \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Or if we demonstrate it for a $n \times n \times n$ cubes whose *transparency* is affected by the element value in matrix in a way that the central element is fully opaque, we'll have this shape:



How do we calculate the big sum of such matrix?

We can see that for each layer l_i ("layer l_i" being a 2D matrix cut of the 3D matrix where $z = i$), the big sum of the l_i is $l_{i\pm1} \pm n^2$:

(n = 3)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} = 15$$
$$\begin{bmatrix} 2 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 2 \end{bmatrix} = 15 + 3^2$$
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} = 15$$

We can see that if we know what the grand sum of l_1 is, we'll have an idea of the grand sum for the other layers. Let's call the "grand sum of l_i" as S_i for sake of simplicity. Now we want to know $\sum_{i=1}^{n} S_i$.

Here's a python code that generates the matrix and the grand sum, given the value n:

```
n = int(input('n : '))
sum = 0
h = (n+1)/2
cent = n * 1.5 - 0.5
print('h is ', h)
for z in range(1, n+1):
    for y in range(1, n+1):
        for x in range(1, n+1):
            d = abs(x - h) + abs(y - h) + abs(z - h)
            r = int(cent - d)
            sum += r
            print(r, end =' ')
        print()
    print('-'*10)
print('sum : ', sum)
```

Let's take a look at a bigger odd n. For example $n = 7$:

$$
\begin{aligned}
&[\ldots]S_1 = S_1 \\
&[\ldots]S_2 = S_1 + n^2 \\
&[\ldots]S_3 = S_1 + 2n^2 \\
&[\ldots]S_4 = S_1 + 3n^2 \\
&[\ldots]S_5 = S_1 + 2n^2 \\
&[\ldots]S_6 = S_1 + n^2 \\
&[\ldots]S_7 = S_1
\end{aligned}
$$

If we take the center layer l_4 out, we'll have

$$
\begin{pmatrix} S_1 \\ S_1 + n^2 \\ S_1 + 2n^2 \\ \cancel{S_1 + 3n^2} \\ S_1 + 2n^2 \\ S_1 + n^2 \\ S_1 \end{pmatrix} \rightarrow 2 \times \begin{pmatrix} S_1 \\ S_1 + n^2 \\ S_1 + 2n^2 \end{pmatrix}
$$

Which means, if n is odd:

$$
\begin{aligned}
\sum_{i=1}^{n} S_1 &= 2 \times \left(\sum_{i=1}^{\frac{n-1}{2}} S_i \right) + S_{\frac{n+1}{2}} \\
&= 2 \times \left(\sum_{i=1}^{\frac{n-1}{2}} (S_1 + (i-1).n^2) \right) + \left(S_1 + \frac{n-1}{2} n^2 \right) \\
&= 2 \times \left(\frac{n-1}{2} S_1 + \sum_{i=1}^{\frac{n-1}{2}} \left((i-1).n^2 \right) \right) + \left(S_1 + \frac{n-1}{2} n^2 \right) \\
&= (n-1)S_1 + \left(2 \times \sum_{i=1}^{\frac{n-1}{2}} \left((i-1).n^2 \right) \right) + \left(S_1 + \frac{n-1}{2} n^2 \right) \\
&= n.S_1 + \left(2 \times \sum_{i=1}^{\frac{n-1}{2}} \left((i-1).n^2 \right) \right) + \frac{n^3 - n^2}{2}
\end{aligned}
$$

$$= n.S_1 + \left(2n^2 \times \sum_{i=1}^{\frac{n-1}{2}} (i-1)\right) + \frac{n^3 - n^2}{2}$$
$$= n.S_1 + \left(2n^2 \times \sum_{i=1}^{\frac{n-3}{2}} i\right) + \frac{n^3 - n^2}{2}$$
$$= n.S_1 + \left(2n^2 \times \left(\frac{n-1}{2}\right)\left(\frac{n-3}{4}\right)\right) + \frac{n^3 - n^2}{2}$$
$$= n.S_1 + \left(\frac{n^4 - 4n^3 + 3n^2}{4}\right) + \frac{n^3 - n^2}{2}$$
$$= n.S_1 + \frac{n^4 - 2n^3 + n^2}{4}$$

And S_1 is $\frac{n^3+n(n \bmod 2)}{2}$ based on [A317614 - OEIS](), and in case of n being odd, we can conclude $S_1 = \frac{n^3+n}{2}$, so:

$$\sum_{i=1}^{n} S_1$$
$$= n.S_1 + \frac{n^4 - 2n^3 + n^2}{4}$$
$$= \frac{n^4 + n^2}{2} + \frac{n^4 - 2n^3 + n^2}{4}$$
$$= \frac{3n^4 - 2n^3 + 3n^2}{4}$$
$$= \frac{3n^2}{4}\left(n^2 - \frac{2}{3}n + 1\right).$$

Now that the function for every odd n is solved, let's take a look at how it's like when n is even, for example when $n = 6$:

$$[\dots]S_1 = S_1$$
$$[\dots]S_2 = S_1 + n^2$$
$$[\dots]S_3 = S_1 + 2n^2$$
$$[\dots]S_4 = S_1 + 2n^2$$
$$[\dots]S_5 = S_1 + n^2$$
$$[\dots]S_6 = S_1$$

It's symmetric along the central non-existing S , in this case $S_{3.5}$:

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_1 + n^2 \\ S_1 + 2n^2 \\ S_1 + 2n^2 \\ S_1 + n^2 \\ S_1 \end{pmatrix} \to 2 \times \begin{pmatrix} S_1 \\ S_1 + n^2 \\ S_1 + 2n^2 \end{pmatrix}$$

So we can conclude that if n is even, we'll have:

$$\sum_{i=1}^{n} S_i = 2 \times \sum_{i=1}^{\frac{n}{2}} S_i$$
$$= 2 \times \sum_{i=1}^{\frac{n}{2}} (S_1 + (i-1)n^2)$$

$$= 2 \times \left(\frac{n}{2} S_1 + n^2 \sum_{i=1}^{\frac{n}{2}} (i-1) \right)$$

$$= 2 \times \left(\frac{n}{2} S_1 + n^2 \sum_{i=1}^{\frac{n-2}{2}} i \right)$$

$$= 2 \times \left(\frac{n}{2} S_1 + n^2 \times \frac{n}{2} \times \frac{n-2}{4} \right)$$

$$= 2 \times \left(\frac{n}{2} S_1 + \frac{n^4 - 2n^3}{4} \right)$$

$$= n.S_1 + \frac{n^4 - 2n^3}{2}$$

placing $S_1 =$ [A317614 - OEIS :]

$$= \frac{n^4}{2} + \frac{n^4 - 2n^3}{4}$$

$$= \frac{3n^4 - 2n^3}{4}$$

$$= \frac{3}{4} n^2 \left(n^2 - \frac{2}{3} n \right)$$

Hence we can conclude that for any n, it's:

$$\frac{3}{4} n^2 \left(n^2 - \frac{2}{3} n + (n \bmod 2) \right).$$

First 20 elements of this series:

8, 54, 160, 425, 864, 1666, 2816, 4617, 7000, 10406, 14688, 20449, 27440, 36450, 47104, 60401, 75816, 94582, 116000

Here's a little python code that generates the series :

```
import math
max = int(input('range : '))
make_plot = 'y' in input('Make plot?[Y/N]').lower()
def f(n):
        r = 0.75*n**2*(n**2 - 2/3*n + n % 2)
        return int(r)
seq = []
n = max
for n in range(2, max+1):
        sum = f(n)
        seq.append(sum)
        print(sum, end=', ')
if make_plot:
        from matplotlib.pyplot import plot, show
        plot(seq)
        show()
```