The distribution of the distance from the first weak subcedance to 1 on permutations

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Abstract

This note finds the explicit distribution of the statistic "distance from the first weak subcedance to 1" on permutations of $[n]$, sequence A350158 in OEIS.

A subcedance in a permutation is an entry that is less than its position: for a permutation $\pi = \pi_1 \ldots \pi_n$ of $[n]$, we say π_i is a *subcedance* if $\pi_i < i$, and a *weak subcedance* if $\pi_i \le i$. The entry 1 is a weak subcedance no matter what its position, and so the *first* subcedance in a permutation of $[n]$ always occurs weakly to the left of 1. We define the statistic X on $\mathcal{S}_n$, the set of all permutations of $[n]$, to be the difference between the position of 1 and the position of the first weak subcedance. Thus, for $\pi = 132$, $X = 0$ while for $\pi = 3214$, $X = 1$ and for $\pi = 246351$, $X = 2$. For $n \ge 1$ and $k \ge 0$, set $\mathcal{A}_{n,k} = \{\pi \in \mathcal{S}_n : X = k\}$ and let $a_{n,k} = |\,\mathcal{A}_{n,k}\,|$. The first few values of $a_{n,k}$ are

$n \backslash k$	0	1	2	3	4
1	1				
2	2	0			
3	5	1	0		
4	17	5	2	0	
5	75	23	16	6	0

Short table of values of $a_{n,k}$, sequence A350158.

Our objective is to compute $a_{n,k}$.

Suppose first that $k \ge 1$ (and $n \ge 2$). Then, for $\pi \in \mathcal{A}_{n,k}$, the position j of the first weak subcedance satisfies $j \le n-k$ because 1 is in position $j+k$. So "delete 1 and subtract 1 from all remaining entries" is obviously a bijection from $\mathcal{A}_{n,k}$ to the set of permutations of $[n-1]$ for which the first (strong) subcedance occurs at or before position $n-k$. The complement of this set in $\mathcal{S}_{n-1}$ consists of those permutations for which none of the first $n-k$ entries are subcedances, that is, all are weak excedances, easily seen to be counted by $k^{n-k}(k-1)!$. Hence, for $n \ge 2$ and $1 \le k \le n-1$,

$$a_{n,k} = (n-1)! - k^{n-k}(k-1)!\,.$$

Note that $a_{n,n-1} = 0$ for $n \geq 2$. A routine calculation now yields that, for $n \geq 2$,

$$a_{n,0} = n! - \sum_{k=1}^{n-2} a_{n,k} = 2(n-1)! + \sum_{k=1}^{n-2} k^{n-k}(k-1)!\,,$$

and the summands in the last expression in fact count $\mathcal{A}_{n,0}$ by the position of 1.

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