# Formula for A349593

OEIS A349593: Square array read by downward diagonals: for $n \geq 0, k \geq 1$, $T(n,k)$ is the period of $\left\{\binom{N}{n} \bmod k : N \in \mathbb{Z}\right\}$.

**Theorem.** Given $n \geq 0, k \geq 1$, let $T(n,k)$ be the period of $\left\{\binom{N}{n} \bmod k : N \in \mathbb{Z}\right\}$. Then we have:
(i) If $n > 0$, then $T(n,k) = k \prod_{\text{prime } p|k} p^{[\log_p(n)]}$, where $[\cdot]$ is the floor function;
(ii) Let $Q(N) = \begin{cases} 1, & k|\binom{N}{n} \\ 0, & k \not|\binom{N}{n} \end{cases}$, then $T(n,k)$ is also the period of $\{Q(N) : N \in \mathbb{Z}\}$.

*Proof.* In the proof we would introduce the following notations:

- $T'(n,k)$ is the period of $\{Q(N) : N \in \mathbb{Z}\}$;
- $s(p) = 1 + [\log_p(n)]$ is the number of digits of $n$ in base $p$;
- $c(N,p)$ is the number of carries when adding $N-n$ and $n$ in base $p$.

Write $k = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, then it is clear that $T(n,k) = \operatorname{lcm}(T(n,p_1^{e_1}), T(n,p_2^{e_2}), \cdots, T(n,p_r^{e_r}))$, and that $T'(n,k)|T(n,k)$. We will complete the proof by proving these two statements: (a) $T(n,p^e) \leq p^{s(p)+e-1}$; (b) $T'(n,k) \geq \prod_{i=1}^{r} p_i^{s(p_i)+e_i-1}$. Using the following Lemma, we only need to consider the case $N \geq n$.

**Lemma.** Let $T(n,k)$ and $T'(n,k)$ be the periods of $\left\{\binom{N}{n} \bmod k : N \geq n\right\}$, $\{Q(N) : N \geq n\}$ respectively. Then $T(n,k)$ and $T'(n,k)$ are also periods of $\left\{\binom{N}{n} \bmod k : N \in \mathbb{Z}\right\}$, $\{Q(N) : N \in \mathbb{Z}\}$ respectively.

*Proof.* For $N \in \mathbb{Z}$, let $r(N)$ be the number in $[n, n+T(n,k))$ that is congruent to $N$ modulo $T(n,k)$. It suffices to show $\binom{N}{n} \equiv \binom{r(N)}{n} \pmod{k}$ holds for all $N \in \mathbb{Z}$. Note that for $N \in \mathbb{Z}$, we have

$$\binom{N+k\cdot n!}{n} = \frac{(N+k\cdot n!)(N+k\cdot n!-1)\cdots(N+k\cdot n!-n+1)}{n!} \equiv \frac{N(N-1)\cdots(N-n+1)}{n!} = \binom{N}{n} \pmod{k}.$$

For $N \in \mathbb{Z}$, choose a sufficiently large $t$ such that $N + t\cdot T(n,k)\cdot n! \geq n$, then

$$\binom{N}{n} \equiv \binom{N+t\cdot T(n,k)\cdot n!}{n} \equiv \binom{r(N+t\cdot T(n,k)\cdot n!)}{n} = \binom{r(N)}{n} \pmod{k}.$$

□

Proof of (a): by Theorem 1 of the Andrew Granville's link, for $N \geq n$, we have

$$\binom{N}{n} / p^{c(N,p)} \equiv \binom{N+p^{s(p)+e-1}}{n} / p^{c(N+p^{s(p)+e-1},p)} \pmod{p^e},$$

so it suffices to show $\min\{c(N,p), e\} = \min\{c(N+p^{s(p)+e-1},p), e\}$. Consider the case where $c(N,p) \neq c(N+p^{s(p)+e-1},p)$, this means that there is a carry when adding $N-n$ and $n$ or adding $N+p^{s(p)+e-1}-n$ and $n$ (in base $p$) beyond or on the $(s(p)+e-1)-$th digit. Suppose that there is a carry when adding $N-n$ and $n$ beyond or on the $(s(p)+e-1)-$th digit, since $n$ only has $s(p)$ digits, there are carries on the $s(p)-1, s(p), \cdots, (s(p)+e-2)-$th digits when adding $N-n$ and $n$. Since $N-n$ and $N+p^{s(p)+e-1}-n$

have the same last $s(p)+e-1$ digits, there are carries on the $s(p)-1, s(p), \cdots, (s(p)+e-2)-$th digits when adding $N+p^{s(p)+e-1}-n$ and $n$, hence $c(N,p), c(N+p^{s(p)+e-1},p) \geq e$. The same follows if there is a carry when adding $N+p^{s(p)+e-1}-n$ and $n$ beyond or on the $(s(p)+e-1)-$th digit. In conclusion, $p^{s(p)+e-1}$ is a period of $\left\{\binom{N}{n} \bmod k : N \geq n\right\}$.

Proof of (b): Let $P=\prod_{i=1}^{r} p_i^{s(p_i)+e_i-1}$, by (a) we have $T'(n,k)|P$. Fixed $i$, we will show that $P/p_i$ is not a period of $\{Q(N): N \geq n\}$. For each $p_j$, consider the base$-p_j$ expansion of $n$: $n = n_j p_j^{s(p_j)-1} + \cdots$, $n_j \neq 0$. Let $N \geq n$ satisfy the congruence

$$N-n \equiv (p_j-n_j)p_j^{s(p_j)-1} + (p-1)p_j^{s(p_j)} + \cdots + (p-1)p_j^{s(p_j)+e_j-2} \big(\bmod\ p_j^{s(p_j)+e_j-1}\big),$$

then there are at least $e_j$ carries when adding $N-n$ and $n$ in base $p_j$; by Kummer's theorem, $k|\binom{N}{n}$. Let $r \geq 0$ satisfy the congruence

$$\begin{cases} r \equiv 0 \big(\bmod\ p_j^{s(p_j)+e_j-1}\big), j \neq i; \\ r \equiv -p_i^{s(p_i)+e_i-2} \big(\bmod\ p_i^{s(p_i)+e_i-1}\big), \end{cases}$$

then

$$N+r-n \equiv \begin{cases} (p_i-n_i)p_i^{s(p_i)-1} + (p-1)p_i^{s(p_i)} + \cdots + (p-1)p_i^{s(p_i)+e_i-3} + (p-2)p_i^{s(p_i)+e_i-2}, e_i > 1 \\ (p_i-n_i-1)p_i^{s(p_i)-1}, e_i = 1 \end{cases}$$
$$\big(\bmod\ p_i^{s(p_i)+e_i-1}\big),$$

there are only $e_i-1$ carries when adding $N+r-n$ and $n$ in base $p_i$, hence $k \not| \binom{N+r}{n}$. Note that $P/p_i|r$, so $P/p_i$ is not a period of $\{Q(N): N \geq n\}$. □