WALKS OF UP AND RIGHT STEPS IN THE SQUARE LATTICE WITH BLOCKED SQUARES

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Abstract. We count walks on the square lattice consisting of Up and Right steps starting at the origin and ending at some predefined integer coordinate (x, y). The standard walks are counted by binomial coefficients. We look at restricted paths that avoid a regular subset of squares in the sense that a subset of squares at coordinates $(x, y) = (r, r) \pmod 3$ is avoided (blocked, inaccessible) and compute bivariate generating functions for the number of these restricted walks for two examples.

1. Introduction

We consider walks on the (simple) square lattice starting at the origin $(x, y) = (0, 0)$, consisting of U(p) and R(ight) steps, and ending at a generic (x, y) coordinate. Because the path contains only R and U steps, the walks are obviously self-avoiding and remain in the first quadrant A simple statistics is the number of walks $W_{x,y}$ from $(0, 0)$ to (x, y). The total number of steps is $x + y$, and the x R and y U steps may be executed in any order. The binomial coefficients of the Pascal triangle ensue:

$$W_{x,y} = \binom{x+y}{x}. \tag{1}$$

The bivariate generating function is obtained by re-summation of the double sum yielding a geometric series:

$$W(u,v) = \sum_{x,y\ge 0} W_{x,y} u^x v^y = \sum_{x\ge 0}\sum_{y\ge 0}\binom{x+y}{x} u^x v^y = \sum_{t\ge 0}\sum_{x=0}^{t}\binom{t}{x} u^x v^{t-x} \tag{2}$$

$$= \sum_{t\ge 0} v^t \sum_{x=0}^{t}\binom{t}{x}(u/v)^x = \sum_{t\ge 0} v^t (1+\frac{u}{v})^t = \sum_{t\ge 0}(v+u)^t = \frac{1}{1-v-u}.$$

2. Blocked "odd" squares

A modified version of the walks may consider steps onto squares blocked/inaccessible where either x or y are odd numbers. Figure 1 is an example of 2 paths from $(0, 0)$ to $(2, 4)$ where the blocked squares are dark:

We may define a number of walks $W^{(n,m_1,m_2,\ldots)}$ as the number of walks where the squares at $(x, y) = (jn + m_1, kn + m_1)$ or $(x, y) = (jn + m_2, kn + m_2)$ etc are inaccessible; that means subdividing the square lattice into regular $n \times n$ tiles

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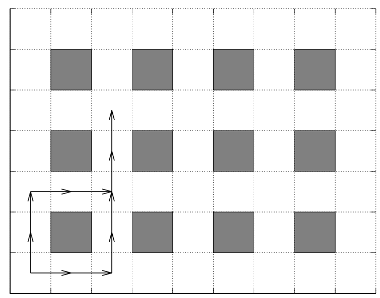

FIGURE 1. Example of paths in the $W^{(2,1)}$ Grid

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(or unit cells), we consider a subset of squares at coordinates (m_1, m_1), (m_2, m_2) within the tiles inaccessible.

Let $W^{(n,m_1,\ldots)}_{j,k,s,t}$ count the paths from $(0, 0)$ to $(x, y) = (jn + s, kn + t)$. The recursive approach to calculate the number of these paths is to accumulate the paths to the left and down neighbors of these squares, and to skip the contributions if these neighbors are inaccessible. For the case of 2×2 tiles where 1 square of both coordinates odd is inaccessible:

$$W^{(2,1)}_{j,k,0,0} = W^{(2,1)}_{j,k-1,0,1} + W^{(2,1)}_{j-1,k,1,0}; \tag{3}$$

$$W^{(2,1)}_{j,k,0,1} = W^{(2,1)}_{j,k,0,0}; \tag{4}$$

$$W^{(2,1)}_{j,k,1,0} = W^{(2,1)}_{j,k,0,0}; \tag{5}$$

$$W^{(2,1)}_{j,k,1,1} = 0. \tag{6}$$

Bivariate generating functions are

$$W^{(n,m_1,\ldots)}_{s,t}(u, v) \equiv \sum_{j,k\geq 0} W^{(n,m_1,\ldots)}_{j,k,s,t} u^j v^k. \tag{7}$$

Multiplying (3)–(6) by $u^j v^k$ and summing over j and k transforms these to

$$W^{(2,1)}_{0,0}(u, v) = uW^{(2,1)}_{0,1}(u, v) + vW^{(2,1)}_{1,0}(u, v) + W^{(2,1)}_{0,0,0,0}; \tag{8}$$

$$W^{(2,1)}_{0,1}(u, v) = W^{(2,1)}_{0,0}(u, v); \tag{9}$$

$$W^{(2,1)}_{1,0}(u, v) = W^{(2,1)}_{0,0}(u, v); \tag{10}$$

$$W^{(2,1)}_{1,1}(u, v) = 0. \tag{11}$$

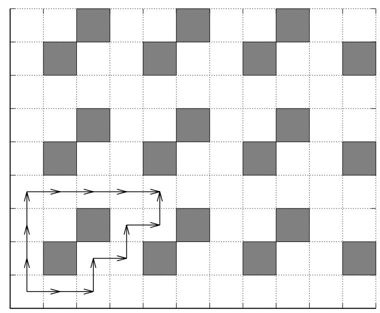

FIGURE 2. Example of 2 paths out of $W^{(3,1,2)}_{1,1,1,0} = 6$ from $(0,0)$ to $(4,3)$ in the $W^{(3,1,2)}$ Grid.

The constant term in (8) appears because (3) is only valid for $j,k \geq 1$. Insertion of the last three equations into the first yields

$$W^{(2,1)}_{0,0}(u,v) = uW^{(2,1)}_{0,0}(u,v) + vW^{(2,1)}_{0,0}(u,v) + W^{(2,1)}_{0,0,0,0}. \tag{12}$$

This type of equation is an eigenvalue equation which defines W up to a constant factor.

$$(1-u-v)W^{(2,1)}_{0,0}(u,v) = W^{(2,1)}_{0,0,0,0}. \tag{13}$$

This ends up with the generating function already known as (2). This is expected, because the regions blocked in Fig. 1 leave exactly as many sub-paths to reach coordinate $(s,t) = (0,0)$ from adjacent tiles as in the case without blocks: either clockwise or counter-clockwise around a block.

3. BLOCKED "MOD 3" SQUARES

3.1. Two blocked along tile diagonals. Another case of paths with two squares in 3×3 tiles blocked is shown in Figure 2. Inaccessible squares are defined here when $(x,y) \equiv (1,1) \pmod 3$ or $(x,y) \equiv (2,2) \pmod 3$.

For the 9 sub-squares in each 3×3 tile the recurrences are

$$W^{(3,1,2)}_{j,k,0,0} = W^{(3,1,2)}_{j,k-1,0,2} + W^{(3,1,2)}_{j-1,k,2,0}; \tag{14}$$

$$W^{(3,1,2)}_{j,k,0,1} = W^{(3,1,2)}_{j,k,0,0} + W^{(3,1,2)}_{j-1,k,2,1}; \tag{15}$$

$$W^{(3,1,2)}_{j,k,0,2} = W^{(3,1,2)}_{j,k,0,1}; \tag{16}$$

$$W^{(3,1,2)}_{j,k,1,0} = W^{(3,1,2)}_{j,k-1,1,2} + W^{(3,1,2)}_{j,k,0,0}; \tag{17}$$

$$W^{(3,1,2)}_{j,k,1,1} = 0. \tag{18}$$

$$W^{(3,1,2)}_{j,k,1,2} = W^{(3,1,2)}_{j,k,0,2}; \tag{19}$$

$$W^{(3,1,2)}_{j,k,2,0} = W^{(3,1,2)}_{j,k,1,0}; \tag{20}$$

$$W^{(3,1,2)}_{j,k,2,1} = W^{(3,1,2)}_{j,k,2,0}; \tag{21}$$

$$W^{(3,1,2)}_{j,k,2,2} = 0. \tag{22}$$

which translates to a linear system of equations for the generating functions:

$$W^{(3,1,2)}_{0,0} = vW^{(3,1,2)}_{0,2} + uW^{(3,1,2)}_{2,0} + W^{(3,1,2)}_{0,0,0,0}; \tag{23}$$

$$W^{(3,1,2)}_{0,1} = W^{(3,1,2)}_{0,0} + uW^{(3,1,2)}_{2,1} + W^{(3,1,2)}_{0,0,0,1}; \tag{24}$$

$$W^{(3,1,2)}_{0,2} = W^{(3,1,2)}_{0,1}; \tag{25}$$

$$W^{(3,1,2)}_{1,0} = vW^{(3,1,2)}_{1,2} + W^{(3,1,2)}_{0,0} + W^{(3,1,2)}_{0,0,1,0}; \tag{26}$$

$$W^{(3,1,2)}_{1,1} = 0. \tag{27}$$

$$W^{(3,1,2)}_{1,2} = W^{(3,1,2)}_{0,2}; \tag{28}$$

$$W^{(3,1,2)}_{2,0} = W^{(3,1,2)}_{1,0}; \tag{29}$$

$$W^{(3,1,2)}_{2,1} = W^{(3,1,2)}_{2,0}; \tag{30}$$

$$W^{(3,1,2)}_{2,2} = 0. \tag{31}$$

Elimination of the variables with no or one term on the right hand sides reduces this to 3 equations:

$$W^{(3,1,2)}_{0,0} = vW^{(3,1,2)}_{0,1} + uW^{(3,1,2)}_{1,0} + W^{(3,1,2)}_{0,0,0,0}; \tag{32}$$

$$W^{(3,1,2)}_{0,1} = W^{(3,1,2)}_{0,0} + uW^{(3,1,2)}_{1,0} + W^{(3,1,2)}_{0,0,0,1}; \tag{33}$$

$$W^{(3,1,2)}_{1,0} = vW^{(3,1,2)}_{0,1} + W^{(3,1,2)}_{0,0} + W^{(3,1,2)}_{0,0,1,0}. \tag{34}$$

This has the format

$$\vec{W} = X\cdot\vec{W} + \hat{W} \tag{35}$$

for a vector $\vec{W} = (W^{(3,1,2)}_{0,0}, W^{(3,1,2)}_{0,1}, W^{(3,1,2)}_{1,0})$ with the matrix

$$X \equiv \begin{pmatrix} 0 & v & u \\ 1 & 0 & u \\ 1 & v & 0 \end{pmatrix} \tag{36}$$

where $\hat{W}$ specifies on which square of the tile at the origin the walk starts. The equation $(1-X)\cdot\vec{W}(u,v) = \hat{W}$ with 1 meaning the unit matrix is solved using

$j\backslash k$	0	1	2	3	4						
0	1										
1	1	4									
2	1	8	28								
3	1	12	64	212							
4	1	16	116	520	1676						
5	1	20	184	1052	4288	13604					
6	1	24	268	1872	9316	35784	112380				
7	1	28	368	3044	17976	81708	301440	940020			
8	1	32	484	4632	31740	167376	713940	2558280	7936620		
9	1	36	616	6700	52336	314932	1531000	6231100	21842560	67494980	
10	1	40	764	9312	81748	553688	3029484	13853584	54389444	187412104	577309148

TABLE 1. The number of paths $W^{(3,1,2)}_{j,k,0,0}$ of reaching $(x,y) = (3j,3k)$ starting at $(0,0)$ in lattice of Figure 2 generated by (38). The array is symmetric with respect to $j \leftrightarrow k$, so only the non-redundant values for $k \le j$ are printed.

matrix inversion,

$$\vec{W} = (1-X)^{-1}\hat{W} = \frac{1}{1-u-v-3uv}\begin{pmatrix} 1-uv & v(1+u) & u(v+1) \\ 1+u & 1-u & 2u \\ 1+v & 2v & 1-v \end{pmatrix}\hat{W}. \tag{37}$$

If we consider paths starting at $(0,0)$, the anchor is $\hat{W} = (1,0,0)$, and the first column of the (inverted) matrix is the bivariate generating function for the paths reaching the various tiles. In particular

$$\frac{1-uv}{1-u-v-3uv} = 1+u+v+u^2+4uv+v^2+u^3+8u^2v+8uv^2+v^3+\cdots \tag{38}$$

is the generating function for walks reaching $(0,0)$ in tiles (j,k). The array of coefficients is shown in Table 1. The diagonal is [1, A085363]. Also

$$\frac{1+u}{1-u-v-3uv} = 1+2u+v+2u^2+6uv+v^2+2u^3+14u^2v+10uv^2+v^3+\cdots \tag{39}$$

is the generating function for walks reaching $(1,0)$ in tiles (j,k). The array of coefficients is shown in Table 2 and appears in [1, A307584]; its sub-diagonal where $k = j+1$ is [1, A051708].

3.2. **One square blocked along tile diagonals.** The lattice where only the center square of each 3×3 tile is blocked is handled equivalently to the previous section. Inaccessible squares are defined here when $(x,y) \equiv 1 \pmod 3$. The recurrences for

$j\backslash k$	0	1	2	3	4				
0	1	2	2	2	2	2	2	2	2
1	1	6	14	22	30	38	46	54	62
2	1	10	42	106	202	330	490	682	906
3	1	14	86	318	838	1774	3254	5406	8358
4	1	18	146	722	2514	6802	15378	30546	55122
5	1	22	222	1382	6062	20406	56190	132870	279630
6	1	26	314	2362	12570	51162	168570	470010	1148250
7	1	30	422	3726	23382	112254	434310	1410030	3968310
8	1	34	546	5538	40098	222498	993570	3706530	11904930
9	1	38	686	7862	64574	407366	2068430	8755670	31780190
10	1	42	842	10762	98922	700010	3990538	18951498	76998698

TABLE 2. The number of paths $W^{(3,1,2)}_{j,k,1,0}$ of reaching $(x,y) = (3j+1, 3k)$ starting at $(0,0)$ in the lattice of Figure 2 generated by (39).

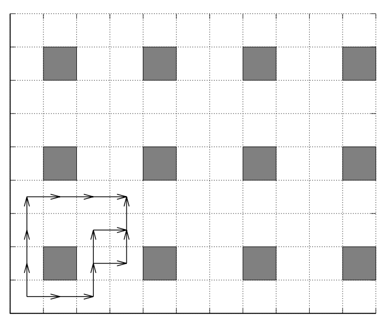

FIGURE 3. Example of 3 paths out of $W^{(3,1)}_{1,1,0,0} = 8$ from $(0,0)$ to $(3,3)$ in the $W^{(3,1)}$ Grid.

the 9 squares in each 3×3 tile are

(40) $$W^{(3,1)}_{j,k,0,0} = W^{(3,1)}_{j,k-1,0,2} + W^{(3,1)}_{j-1,k,2,0};$$

(41) $$W^{(3,1)}_{j,k,0,1} = W^{(3,1)}_{j,k,0,0} + W^{(3,1)}_{j-1,k,2,1};$$

(42) $$W^{(3,1)}_{j,k,0,2} = W^{(3,1)}_{j,k,0,1} + W^{(3,1)}_{j-1,k,2,2};$$

(43) $$W^{(3,1)}_{j,k,1,0} = W^{(3,1)}_{j,k-1,1,2} + W^{(3,1)}_{j,k,0,0};$$

(44) $$W^{(3,1)}_{j,k,1,1} = 0.$$

(45) $$W^{(3,1)}_{j,k,1,2} = W^{(3,1)}_{j,k,0,2};$$

(46) $$W^{(3,1)}_{j,k,2,0} = W^{(3,1)}_{j,k-1,2,2} + W^{(3,1)}_{j,k,1,0};$$

(47) $$W^{(3,1)}_{j,k,2,1} = W^{(3,1)}_{j,k,2,0};$$

(48) $$W^{(3,1)}_{j,k,2,2} = W^{(3,1)}_{j,k,2,1} + W^{(3,1)}_{j,k,1,2}.$$

which translates to the linear system of equations

$$W_{0,0}^{(3,1)} = vW_{0,2}^{(3,1)} + uW_{2,0}^{(3,1)} + W_{0,0,0,0}^{(3,1)}; \tag{49}$$

$$W_{0,1}^{(3,1)} = W_{0,0}^{(3,1)} + uW_{2,1}^{(3,1)} + W_{0,0,0,1}^{(3,1)}; \tag{50}$$

$$W_{0,2}^{(3,1)} = W_{0,1}^{(3,1)} + uW_{2,2}^{(3,1)} + W_{0,0,0,2}^{(3,1)}; \tag{51}$$

$$W_{1,0}^{(3,1)} = vW_{1,2}^{(3,1)} + W_{0,0}^{(3,1)} + W_{0,0,1,0}^{(3,1)}; \tag{52}$$

$$W_{1,1}^{(3,1)} = 0. \tag{53}$$

$$W_{1,2}^{(3,1)} = W_{0,2}^{(3,1)}; \tag{54}$$

$$W_{2,0}^{(3,1)} = vW_{2,2}^{(3,1)} + W_{1,0}^{(3,1)} + W_{0,0,2,0}^{(3,1)}; \tag{55}$$

$$W_{2,1}^{(3,1)} = W_{2,0}^{(3,1)}; \tag{56}$$

$$W_{2,2}^{(3,1)} = W_{2,1}^{(3,1)} + W_{1,2}^{(3,1)}. \tag{57}$$

Elimination of the variables with no or one term on the right hand sides reduces this to :

$$W_{0,0}^{(3,1)} = vW_{0,2}^{(3,1)} + uW_{2,0}^{(3,1)} + W_{0,0,0,0}^{(3,1)}; \tag{58}$$

$$W_{0,1}^{(3,1)} = W_{0,0}^{(3,1)} + uW_{2,0}^{(3,1)} + W_{0,0,0,1}^{(3,1)}; \tag{59}$$

$$W_{0,2}^{(3,1)} = W_{0,1}^{(3,1)} + uW_{2,2}^{(3,1)} + W_{0,0,0,2}^{(3,1)}; \tag{60}$$

$$W_{1,0}^{(3,1)} = vW_{0,2}^{(3,1)} + W_{0,0}^{(3,1)} + W_{0,0,1,0}^{(3,1)}; \tag{61}$$

$$W_{2,0}^{(3,1)} = vW_{2,2}^{(3,1)} + W_{1,0}^{(3,1)} + W_{0,0,2,0}^{(3,1)}; \tag{62}$$

$$W_{2,2}^{(3,1)} = W_{2,0}^{(3,1)} + W_{0,2}^{(3,1)}. \tag{63}$$

This has the format

$$\vec{W} = X \cdot \vec{W} + \hat{W} \tag{64}$$

for a vector $\vec{W} = (W_{0,0}^{(3,1)}, W_{0,1}^{(3,1)}, W_{0,2}^{(3,1)}, W_{1,0}^{(3,1)}, W_{2,0}^{(3,1)}, W_{2,2}^{(3,1)})$ with the matrix

$$X \equiv \begin{pmatrix} 0 & 0 & v & 0 & u & 0 \\ 1 & 0 & 0 & 0 & u & 0 \\ 0 & 1 & 0 & 0 & 0 & u \\ 1 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & v \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{65}$$

The equation $(1 - X) \cdot \vec{W}(u, v) = \hat{W}$ with 1 meaning the unit matrix is solved using matrix inversion,

$$\vec{W} = (1 - X)^{-1}\hat{W} = \frac{1}{1 - 2u - 2v - 7uv + u^2 + v^2} \tag{66}$$

$$\times \left(\begin{array}{cccccc} 1 - u - v - 3uv & v(1 - v + 2u) & v(1 - v + 2u) & u(1 - u + 2v) & u(1 - u + 2v) & uv(2 + \\ 1 - v - uv - u^2 & 1 - 2u - v - 2uv + u^2 & v(1 - v + 5u) & u(v + 2 - 2u) & u(v + 2 - 2u) & 3uv(1 \\ 1 - v + 2u & 1 - v - u & 1 - v - u & 3u & 3u & u(1 - u \\ 1 - u - uv - v^2 & v(2 + u - 2v) & v(2 + u - 2v) & 1 - u - 2v - 2uv + v^2 & u(1 - u + 5v) & 3uv(1 \\ 1 - u + 2v & 3v & 3v & 1 - u - v & 1 - u - v & v(1 - v \\ 2 + u + v & 1 - u + 2v & 1 - u + 2v & 1 - v + 2u & 1 - v + 2u & 1 - u - \end{array} \right.$$

$j\backslash k$	0	1	2	3	4				
0	1								
1	1	8							
2	1	24	150						
3	1	49	541	3116					
4	1	83	1440	12275	68032				
5	1	126	3171	37730	282003	1528776			
6	1	178	6139	97482	948909	6549906	35003238		
7	1	239	10830	221628	2730615	23453001	153481299	812041860	
8	1	309	17811	456897	6959700	72932904	575036475	3622111560	19022666310
9	1	388	27730	871915	16103164	202465120	1895949055	14046869575	85975792075

TABLE 3. The number of paths $W^{(3,1)}_{j,k,0,0}$ of reaching $(x,y) = (3j,3k)$ starting at $(0,0)$ in the lattice of Fig. 3 generated by (67). The array is symmetric with respect to $j \leftrightarrow k$.

If we consider paths starting at $(0,0)$, the anchor is $\hat{W} = (1,0,0,0,0,0)$, and the first column of the (inverted) matrix is the bivariate generating function for the paths reaching the various tiles. In particular

(67)
$$\frac{1-u-v-3uv}{1-2u-2v-7uv+u^2+v^2} = 1+u+v+u^2+8uv+v^2+u^3+24u^2+24uv^2+v^3+\cdots$$

is the generating function for walks reaching $(0,0)$ in tiles (j,k). The array of coefficients is shown in Table 3.

References

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URL: `https://www.mpia.de/~mathar`

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