ON A SUM INVOLVING FACTORIALS

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Let $0 \leq k \leq n$ be two nonnegative integers. Let

$$T(n,k) = \frac{1}{k!}\sum_{i=k}^{n} i!,$$

$$p_n(x) = \sum_{k=0}^{n} T(n,k)x^k.$$

We prove the following results, establishing the three conjectures made in A348482.

Theorem 1. *(1) For $n \geq 1$, we have*

$$T(n,1) = \sum_{k=0}^{n-1} \frac{T(n,k)}{k+2}. \tag{1}$$

(2) For $n \geq 1$, we have

$$p_n(x) = 1 + (x+1)p_{n-1}(x) + \sum_{i=1}^{n-1}\left(\frac{d}{dx}\right)^i p_{n-1}(x). \tag{2}$$

(3) For $n \geq 0$ we have

$$\sum_{k=0}^{n}(-1)^k T(n,k) = \text{A173184}(n). \tag{3}$$

Proof. (1) First, notice that (1) is equivalent to

$$\sum_{i=0}^{n} i! = 1 + \sum_{k=0}^{n-1} \frac{T(n,k)}{k+2}. \tag{4}$$

Setting $a_n = \sum_{i=0}^{n} i!$, we clearly have $a_n = a_{n-1} + n!$. Since equality holds in (4) for $n = 1$, it suffices to show that the right-hand side of (4) also satisfies this

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recursion. We have

$$
\begin{aligned}
1+\sum_{k=0}^{n-1}\frac{T(n,k)}{k+2} = 1+\sum_{k=0}^{n-2}\frac{T(n-1,k)}{k+2}+n! &\iff \\
\sum_{k=0}^{n-2}\frac{T(n,k)-T(n-1,k)}{k+2} = n!-1 &\iff \\
\sum_{k=0}^{n-2}\frac{1}{k!(k+2)} = 1-\frac{1}{n!} &\iff \\
\sum_{k=1}^{n-1}\frac{k}{(k+1)!} = 1-\frac{1}{n!} &\iff \\
\sum_{k=1}^{n-1}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right) = 1-\frac{1}{n!}&.
\end{aligned}
$$

Clearly, the last equality holds true and the assertion follows.

(2) We have

$$
\begin{aligned}
&1+(x+1)p_{n-1}(x)+\sum_{i=1}^{n-1}\frac{d}{dx^i}p_{n-1}(x) &=\\
&1+(x+1)p_{n-1}(x)+\sum_{i=1}^{n-1}\sum_{k=i}^{n-1}\frac{k!}{(k-i)!}T(n-1,k)x^{k-i} &=\\
&1+\sum_{i=0}^{n-1}i!+\sum_{k=1}^{n-1}\left(T(n-1,k)+T(n-1,k-1)\right)x^k+x^n+\sum_{j=1}^{n-1}j!\sum_{k=1}^{j}\sum_{i=1}^{k}\frac{x^{k-i}}{(k-i)!} &=\\
&\sum_{i=0}^{n-1}i!+\sum_{k=0}^{n}x^k+\sum_{k=1}^{n-1}\sum_{i=k}^{n-1}\frac{i!}{k!}(k+1)x^k+\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\frac{k!}{i!}(k-i)x^i &=\\
&p_n(x)+\sum_{k=0}^{n-1}\left(1-\frac{n!}{k!}\right)x^k+\sum_{k=0}^{n-2}\frac{1}{k!}\sum_{i=k+1}^{n-1}i!x^{k+1}+\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\frac{k!}{i!}(k-i)x^i &=\\
&p_n(x)+\sum_{k=0}^{n-1}\left(1-\frac{n!}{k!}\right)x^k+\sum_{k=0}^{n-1}\sum_{i=k}^{n-1}\frac{i!}{k!}ix^k &=\\
&p_n(x)+\sum_{k=0}^{n-1}\left(1-\frac{n!}{k!}+\frac{1}{k!}\sum_{i=k}^{n-1}\left((i+1)!-i!\right)\right)x^k = p_n(x).
\end{aligned}
$$

(3) Let $b_n = \sum_{k=0}^{n}(-1)^k T(n,k)$. We immediately verify that (3) holds for $n = 0, 1, 2$. Due to the recurrence that A173184 satisfies, it suffices to show that

$$b_{n+3} - (n+3)b_{n+2} + (n+2)b_n = 0.$$

We have

$$\begin{aligned}
&b_{n+3} - (n+3)b_{n+2} + (n+2)b_n = \\
&\sum_{k=0}^{n+3}(-1)^k T(n+3,k) - (n+3)\sum_{k=0}^{n+2}(-1)^k T(n+2,k) + (n+2)\sum_{k=0}^{n}(-1)^k T(n,k) = \\
&\sum_{k=0}^{n+3}(-1)^k \frac{1}{k!}\sum_{i=k}^{n+3} i! - (n+3)\sum_{k=0}^{n+2}(-1)^k \frac{1}{k!}\sum_{i=k}^{n+2} i! + (n+2)\sum_{k=0}^{n}(-1)^k \frac{1}{k!}\sum_{i=k}^{n} i! = \\
&\sum_{k=n+1}^{n+3}(-1)^k \frac{1}{k!}\sum_{i=k}^{n+3} i! + \sum_{k=0}^{n}(-1)^k \frac{1}{k!}\sum_{i=n+1}^{n+3} i! - (n+3)\sum_{k=n+1}^{n+2}(-1)^k \frac{1}{k!}\sum_{i=k}^{n+2} i! \\
&\qquad - (n+3)\sum_{k=0}^{n}(-1)^k \frac{1}{k!}\sum_{i=n+1}^{n+2} i! = \\
&(-1)^{n+1} + (-1)^{n+1}(n+3)(n+1) - (n+2)\left((-1)^{n+1}(n+3) - (-1)^{n+1}\right) = 0.
\end{aligned}$$

□

References

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