Agathokakological Numbers
PROOFS

Note: Scientific notation is used frequently in the following proofs. This uses the general form $a=b*10^c$ (variables will change), with variable "a" being any positive integer, the lead variable "b" always having value $1\leq a<10$, and "r" being an integer with $c\geq 0$.

Proof for no k with a lead 9

Take a number w. The number of digits in the juxtaposition of the prime factors of w is d_f and the number of digits in the product of these prime factors is d_p (digits in w). For any number, $d_p \leq d_f$. Number w has prime factors a_1, ..., a_m. Consider these prime factors in the form $a_1=b_1*10^{r-1}$, …, $a_m=b_m*10^{r-m}$. From the definition for Agathokakological Numbers (A.N.'s), $d_p=d_f$, and for $d_p=d_f$, $b_1*...b_m \geq 10^{m-1}$ must be true. Take the minimum value in the set **B** = $\{b_1, …, b_m\}$, denoting the minimum value in the set **B** as "s". Moreover, for $d_p=d_f$, $b_1*...*b_m = z*10^y$, and z<s. Similarly, $a_1*...*a_m = z*10^x$ which retains the same value for z, so z<s. Now, consider $w = L*10^p$ with $9\leq L<10$ (a number with a lead 9). For $a_1*...*a_m = L*10^p$ with $9\leq L<10$ while holding $d_p=d_f$, $b_1*...*b_m \geq 9*10^{m-1}$. Use equation $b_1*...*b_m = z*10^y$ with 9 as the observed value for z. Next, use z<s to show that the minimum value s in the set **B** is greater than 9 (and less than 10 due to scientific notation form, see **Note**). The value s is the minimum value in the set **B**, so if there are other values in the set **B**, all other values b_{other} in the set **B** hold $s\leq b_{other}<10$ with s>9. Further, $a_1 = b_1*10^{r-1}$, … $a_m = b_m*10^{r-m}$, and since all terms "b" in the set **B** are 9<b<10, all prime factor terms a in the set **A** = $\{a_1, …, a_m\}$ will have the form $b*10^u$ with 9<b<10 (all "a" have a lead 9), since the terms in the set **A** are simply the terms in the set **B** multiplied by some power of 10 that is greater than or equal to zero. The prime factors of an A.N. can share no common digits with the A.N. itself, and since any number with a lead 9 and $d_f=d_p$ is composed entirely of prime factors with lead 9s, there are no A.N.'s with a lead 9.

Proof for no Agathokakological Number ends in 2 or 5.

A number ending in 2 or 5 has at least one 2 or 5 in the prime factorization, respectively. Agathokakological Numbers have no common digits with the juxtaposition of their prime factors, so none end in 2 or 5.

Proof for 10 is only Agathokakological Numbers to end in 0

All numbers other than 10 ending in 0 have prime factors a_1, ..., a_m, 2, and 5. The number of digits in a_1, ..., a_m is d_f. The product of $a_1 * ... * a_m$ = p, and the number of digits in p is d_p. $d_p <= d_f$. Now, multiplying p by the remaining prime factors 2 and 5 makes d_p become $d_p + 1$. This is because any number p multiplied by 2 and 5 (10) is 10p, thus the digits in p will always increase by 1 when multiplied by 2 and 5. Conversely, adding 2 and 5 to the list of prime factors makes d_f become $d_f + 2$. The inequality is now $d_p + 1 < d_f + 2$, so the total digits of the prime factorization of a number ending in 0 that is not 10 will always be greater than the number itself, making these numbers ineligible for the sequence. This concurrently proves 10 is the only Agathokakological Number with 5 as a prime factor. All other numbers with 5 as a factor end in either 5 (see **Proof for no Agathokakological Number ends in 2 or 5**) or 0 (no Agathokakological Numbers ending in 0 after 10).

Proof for Agathokakological Numbers "k" can only be square terms when "k" is of the order 10^m where "m" is odd

When "m" is even, "k" will have an odd number of digits. A square term is the product of a number multiplied by itself, so a square term always has an even number of prime factors. As a result, there will always be an even number of digits for the juxtaposition of the square term's prime factorization. Thus, Agathokakological Numbers that are square terms must be of the order 10^m with m being odd.

Proof for numbers k in sequence written as L*10^m can only be even when 1 <= L < $1.\overline{888}$

Consider a number "j" where j is even and j is written in the form $j=L*10^m$ with $2 \leq L<10$. Number j has prime factors a_1, …, a_m, and 2. Number j is d_p digits long. j/2 is also d_p digits long, and j/2 has prime factors a_1, …, a_m. Now consider $a_1*...*a_m=p$. The number of digits in the juxtaposition of prime factors a_1, …, a_m is d_f, and the number of digits in p is d_p. For any number, $d_p \leq d_f$. Now, return 2 to the list of prime factors. As was previously stated, d_p remains the same (if original j was $j=L*10^m$ where j is even and $2 \leq L<10$), but d_f increases by one because a 2 was added to the list, and 2 is one digit long. $d_p<d_f+1$, but an Agathokakological Number's (A.N.'s) juxtaposition of prime factors must be the same length of the A.N. itself, so even A.N.'s do not exist for $2 \leq L<10$.

Now, consider an even number $j=L*10^m$ with $1.\overline{888} < L < 2$. At least one 9 appears for any j in this form. Also, there is at least one prime factor of 2 for any j in this form because j is even. j/2 is equal to a number "g" in the form $g=h*10^{m-1}$ where $9.\overline{444}<h<10$. Call the number of digits in j "d_p" and the digits in the juxtaposition of the prime factors of j "d_f". An A.N. has $d_p=d_f$. Dividing j by 2 with $1.\overline{888}<L<2$ for $j=L*10^m$ gives a number g with one fewer digit, and the juxtaposition of the prime factorization of g is also one fewer digit than the juxtaposition of the prime factorization of j (one less 2 in prime factorization for g). So, for the formula $d_p=d_f$ for j to hold, the number of digits in g "$d_{p,g}$" must be equivalent to the number of digits in the juxtaposition of the prime factorization of g "$d_{f,g}$". Number g always has a lead 9 (recall first digit of g is $9.\overline{444}<h<10$), and in the **Proof for no k with a lead 9** it was shown for any number with a lead 9 and $d_p=d_f$, all prime factorization terms must also have a lead 9. Therefore, the prime factorization for numbers g where $d_{p,g}=d_{f,g}$ is only made up of primes with a lead 9; thus, the prime factors for j when $j=L*10^m$ with $1.\overline{888}<L<2$ are 2 and other prime factor(s) with lead 9s. Since at least one 9 appears for any j in this form, any j in this form has at least one prime factor with a lead 9, and A.N.'s must share no common digits with their prime factorization, no even numbers j can appear in the list of A.N.'s for j=L*10^m with $1.\overline{888}$<L<2.

Even A.N.'s do not appear for j=L*10^m with 2≤L<10 or for $1.\overline{888}$<L<2, so even A.N.'s only appear for j=L*10^m with 1≤L<$1.\overline{888}$.