In this document we prove that the limit of the ratio of successive terms of the sequence A347913, if it exists, is $\geq 2$. We do this by providing a construction for $\mathrm{C}(\mathrm{n}+1)((\mathrm{n}+1)$ th Catalan number) multisets where there are 2 n elements of the multisets. To do this, we first provide a construction for a desired position called $\mathrm{H}(\mathrm{n})$, then we show how to get $\mathrm{C}(\mathrm{n}+1)$ multisets from $H(n)$. This works because the limiting ratio of the Catalan numbers is 4 .

Proof: $\mathrm{H}(\mathrm{n})$ is the multiset where all the integers with absolute value less than or equal to $n-1$ and the same parity as $n-1$ appear twice. The cardinality of $H(n)$ is $2 n$.

The construction for $\mathrm{H}(\mathrm{n})$ is inductive. To construct $\mathrm{H}(\mathrm{n})$, you construct $\mathrm{H}(\mathrm{n}-1)$ and then you apply the final step $X(n)$.

The construction for $X(n)$ is also inductive. You split all the duplicates simultaneously, then you apply $\mathrm{X}(\mathrm{n}-1)$, then you split all the duplicates simultaneously again. It is easy to prove that these steps result in $H(n)$.

There are n duplicates in $\mathrm{H}(\mathrm{n})$. If you split two adjacent duplicates, you get a new duplicate which you can split. Therefore, the duplicate split patterns come in a zero-one array. A zero-one array is defined as a triangular array of zeros and ones such that if there is a one in a square which is not on the diagonal, , then there has to be a one to the right of it and on top of it. Here is an example of such an array:


We can construct a bijection between these arrays and Dyck paths to prove that these arrays are counted by the Catalan numbers. You start two squares to the left of the bottom-left corner of the array. Then, you walk right whenever you see a one or a blank space. Otherwise, you walk up. When you reach the topmost blank space, one square above the top-right corner, you walk up. For the example give above, the Dyck path is:

|  |  |  |  |  | . |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  | . |
|  |  |  |  |  | . |
|  |  |  |  | . | . |
|  | . | . | . | . |  |
| . | $\cdot$ |  |  |  |  |

Now given the Dyck path, how do you construct the zero-one array? This is our example Dyck path:


Starting two squares to the right of the bottom-left corner, you fill in a zero if the path goes above the square and a one if the path goes through or below the square. This way, you get the original array. The array that you get from this path is:

|  |  |  | 1 |
| :--- | :--- | :--- | :--- |
|  |  | 1 | 1 |
|  | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 |

We got our original array back. This argument proves that the number of positions you can reach in Linee starting with $\mathrm{H}(\mathrm{n})$ is the $(\mathrm{n}+1)$ th Catalan number. Therefore, the limiting ratio of A347913 is at least 2.

