On the Number of Interlacing Triangles of Size n

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1 Introduction

The counting of the number of interlacing triangles is a classic example of an enumeration problem. These combinatorial objects are generalizations of shifted standard Young tableaux of shape $(n, n-1, \ldots, 1)$ in so far as, we relax the condition on being strictly increasing in rows and columns, and replace it with an *interlacing* condition. In this work we layout a construction which reduces the problem to counting the linear extensions of a collection of posets.

2 Background

Definition 2.1. *An* interlacing triangle of rank n *is a triangular array of the integers* $1, \ldots, \binom{n+1}{2}$ *such that there are* n *numbers in the first row,* $n-1$ *numbers in the second row, and so on, subject to the following condition:*

If $a(i,j)$ *denotes the* j*-th number in the* i*-th row then either*

$$a(i-1, j+1) < a(i,j) < a(i-1,j) \text{ or } a(i-1,j) < a(i,j) < a(i-1,j+1)$$

for $1 < i \leq n$ *and* $1 \leq j \leq n-i+1$

In this work, we wish to enumerate all interlacing triangles of a given rank. For the remainder of this paper, we let $\mathscr{T}_n = \{\text{interlacing triangles of rank n}\}$ and

$$t_n = \#\{\text{interlacing triangles of rank n}\}$$

.

Ex 2.1: For $n = 2$, we have the following interlacing triangles of rank 2:

$$\begin{array}{ccc} & 2 & \\ 1 & & 3 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 3 & & 1 \end{array}.$$

So we have $t_2 = 2$.

Ex 2.2: For $n = 3$, we have the following interlacing triangles of rank 3:

$$\begin{array}{ccccc} & & 3 & & \\ & 5 & & 2 & \\ 6 & & 4 & & 1 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 5 & & 2 & \\ 6 & & 3 & & 1 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 5 & & 2 & \\ 3 & & 6 & & 1 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 5 & & 2 & \\ 4 & & 6 & & 1 \end{array}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 5 & & 3 & \\ 2 & & 6 & & 1 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 5 & & 3 & \\ 1 & & 6 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 4 & & 2 & \\ 5 & & 1 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 4 & & 2 & \\ 6 & & 1 & & 5 \end{array}$$

$$\begin{array}{ccccc} & & 3 & & \\ & 5 & & 2 & \\ 6 & & 1 & & 4 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 5 & & 2 & \\ 6 & & 1 & & 3 \end{array}$$

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 4 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 3 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 1 & & 6 & & 3 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 1 & & 6 & & 4 \end{array}$$

$$\begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 1 & & 6 & & 2 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 3 & & 5 & \\ 2 & & 6 & & 1 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 6 & & 1 & & 5 \end{array} \qquad \begin{array}{ccccc} & & 3 & & \\ & 2 & & 4 & \\ 5 & & 1 & & 6 \end{array}$$

$$\begin{array}{ccccc} & & 3 & & \\ & 2 & & 5 & \\ 4 & & 1 & & 6 \end{array} \qquad \begin{array}{ccccc} & & 4 & & \\ & 2 & & 5 & \\ 3 & & 1 & & 6 \end{array}.$$

So we have $t_3 = 20$. Note that each interlacing triangle in the bottom three rows is a reflection of an interleaving triangle in the top three rows over the vertical axis of symmetry.

We now recall some basic definitions from the theory of posets, all of which can be found in [3].

Definition 2.2. *A* partially ordered set *or* poset *is a set P together with a binary relation that is reflexive, antisymmetric, and transitive, denoted $\leq_P$ or $\leq$ if the context is clear. We denote the poset and its binary relation as the pair $(P, \leq)$*

A poset is said to be finite if the underlying set P is a finite set. The posets that are of interest to us are on $\binom{n+1}{2}$ elements for some n.

Definition 2.3. *A* linear extension *of a poset $(P, \leq)$ is an order preserving bijection $\sigma : P \longrightarrow [|P|]$ where $[|P|]$ is given the natural ordering. Let $\mathscr{L}(P)$ be the set of linear extensions on P. Let $\ell(P)$ be the number of linear extensions on P.*

In general, there is no formula for counting linear extensions of a given poset. See [1] for more details on the computational complexity of counting linear extensions. However, there are certain classes of posets which do yield closed formulas for the number of linear extensions. Recent work, by Hopkins [2] elaborates on the connection between product formulas for $\ell(P)$ and properties of the poset's *dynamical behavior* with respect to *promotion* and *rowmotion*.

3 Poset Construction

In this section we lay out a decomposition of {interlacing triangles of rank n} into equivalence classes corresponding to certain posets on $\binom{n+1}{2}$ elements. To begin, we consider a triangular array of dots corresponding to positions (i, j) in a generic interleaving triangle. Now, associate the set $\mathcal{A}(n) = \bigoplus_{i=1}^{\binom{n}{2}} \mathbb{Z}/2\mathbb{Z}$ to the triangular array, in such a way that the first $n-1$ summands are aligned with the second row of the array and the next $n-2$ summands are aligned with the third row of the array, and so on. This reindexes the summands in $\mathcal{A}(n)$. For example, if $n = 4$, we have the triangular array:

$$\begin{array}{ccccccc} & & & \bullet & & & \\ & & \bullet & & \bullet & & \\ & \bullet & & \bullet & & \bullet & \\ \bullet & & \bullet & & \bullet & & \bullet \end{array},$$

and the associated set, $\mathcal{A}(4)$ is represented below,

$$\begin{array}{ccccc} & & \mathbb{Z}/2\mathbb{Z} & & \\ & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & \\ \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \end{array}.$$

An element $x \in \mathcal{A}(n)$ is given by a triangular array of $\binom{n}{2}$ 0's or 1's. For example,

$$\begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 1 \end{array} \quad \in \quad \begin{array}{ccccc} & & \mathbb{Z}/2\mathbb{Z} & & \\ & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & \\ \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} \end{array}.$$

Finally, define

$$\begin{aligned} \mathcal{B}(n) = \{x \in \mathcal{A}(n) : &\text{ if } x(i,j) = 1 \text{ and } x(i-1,j) = x(i-1,j+1) \\ &\text{ then } x(i-1,j) = x(i-1,j+1) \neq 0 \text{ and} \\ &\text{ if } x(i,j) = 0 \text{ and } x(i-1,j) = x(i-1,j+1) \\ &\text{ then } x(i-1,j) = x(i-1,j+1) \neq 1\} \end{aligned}$$

The elements of $\mathcal{B}(n)$ are triangular arrays of 0's and 1's which avoid the following patterns:

$$\begin{array}{ccc} & 0 & \\ 1 & & 1 \end{array} \qquad\qquad \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array}.$$

We refer to such configurations as a $(0,1)$-*configuration*. Its position is defined by the location of the top entry. To illustrate the elements of $\mathcal{B}(n)$ we note that

$$\begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 1 \end{array} \quad \in \quad \mathcal{B}(4),$$

but

$$\begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 0 & & 0 & & 1 \end{array} \qquad \notin \quad \mathcal{B}(4).$$

because there is a $(0,1)$-configuration in position $(3,1)$. Recall that we justify the summands, $\mathbb{Z}/2\mathbb{Z}$, with respect to the triangular array of dots, starting with the second row.

Now construct a *directed graph*, G_x for some $x \in \mathcal{A}(n)$ according to the following rules:

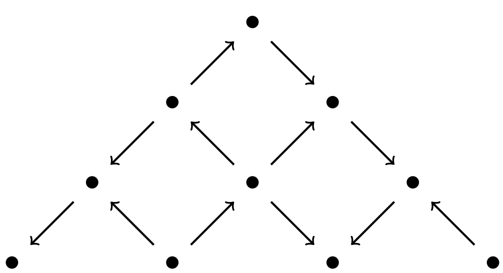
1. For all positions (i,j) such that $x(i,j) = 0$ for $2 \leq i < n$ and $1 \leq j \leq n-i+1$ in the triangular array, connect an edge $(i,j) \longrightarrow (i-1, j+1)$ and $(i,j) \longleftarrow (i-1,j)$.

2. For all positions (i,j) such that $x(i,j) = 1$ for $2 \leq i < n$ and $1 \leq j \leq n-i+1$ in the triangular array, connect an edge $(i,j) \longrightarrow (i-1,j)$ and $(i,j) \longleftarrow (i-1,j+1)$.

Note that G_x is simple. Furthermore, for every edge connecting position (i,j) to a position in row $i-1$, we have an edge leaving (i,j) going to a position in row $i-1$ with index $(i-1,j+1)$ or $(i-1,j)$ depending on whether x has a 0 or a 1 in position (i,j).
Ex 3.1: Consider

$$x = \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 1 \end{array} \qquad \in \quad \mathcal{B}(n),$$

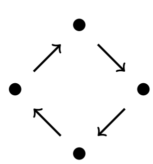
which gives the corresponding directed graph G_x



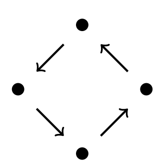
We now present the following Lemma followed by a Proposition.

Lemma 3.1. *G_x has a cycle of length four if and only if x has a $(0,1)$-configuration.*

Proof. Suppose G_x has a cycle of length four. There exists a position in G_x such that:



or



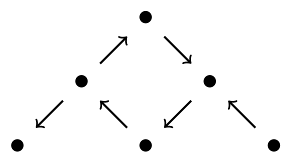
then by the construction of G_x, there is a $(0,1)$-configuration in x:

$$\begin{array}{ccc} & 0 & \\ 1 & & 1 \end{array} \qquad\qquad \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \; .$$

Now suppose (i, j) is a $(0,1)$-configuration in x. We can assume the $(0,1)$-configuration is the following

$$\begin{array}{ccc} & 0 & \\ 1 & & 1 \end{array}$$

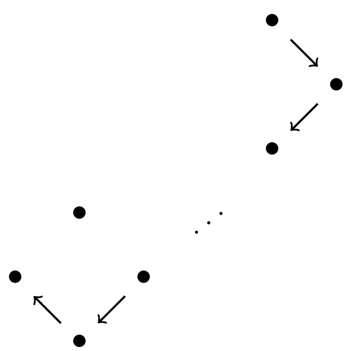
since an analogous argument works for the other type of $(0,1)$-configuration. So by the construction of G_x, we have:



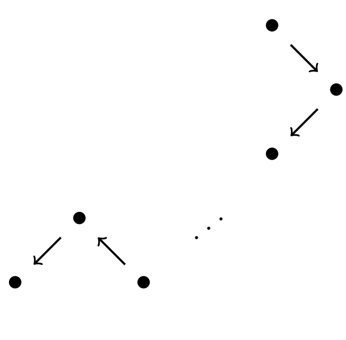
which contains a cycle of length four. □

Proposition 3.1. *$x \in \mathcal{B}(n)$ if and only if G_x is acyclic.*

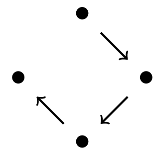
Proof. Assume that there exists a cycle and without loss of generality, it is oriented clockwise. There must be a sequence of consecutive edges SE, SW, and NW, with the possibility of multiple SW edges between the SE and NW edges. So we have following:



Consider the position above the point where the SW edge turns NW. Call this position w. If w corresponds to a 0 in x then we have a cycle of four vertices and so there exists a $(0,1)$-configuration. If not, then we have another cycle which passes through w.



Repeat this process, until we get a sequence SE, SW, NW.



By the construction of G_x, there exists a NE edge, completing a cycle of length four, and so there exists a $(0,1)$-configuration in x and therefore $x \notin \mathcal{B}(n)$.

Now suppose G_x is a directed acyclic graph constructed as above. Assume for contradiction that $x \in \mathcal{A}(n) \setminus \mathcal{B}(n)$. Then there exists a $(0,1)$-configuration which by Lemma 3.1 implies that there exists a cycle of length four in G_x. Therefore G_x is not acyclic. □

To any directed acyclic graph, G, we can associate a poset $(P, \leq_P)$ as follows:

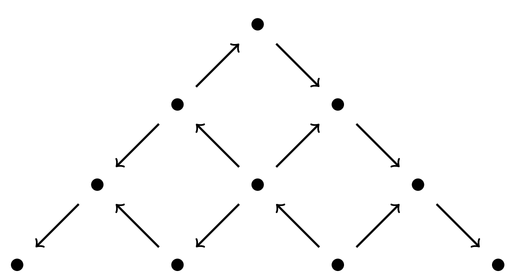
1. Let the underlying set of P be the vertex set of G.

2. For any two vertices u and v in the vertex set of G, we say $u \leq_P v$ if there exists a sequence (possibly empty) of arrows in G that can be traversed starting at u and ending at v.

Clearly, $\leq_P$ is transitive, reflexive, and antisymmetric (the last of which follows from G being acyclic). For $x \in \mathcal{B}(n)$ associate the poset $(P_x, \leq_x)$ by first constructing G_x and then associating the poset P_x to it via the assignment above.
Ex 3.2 Let

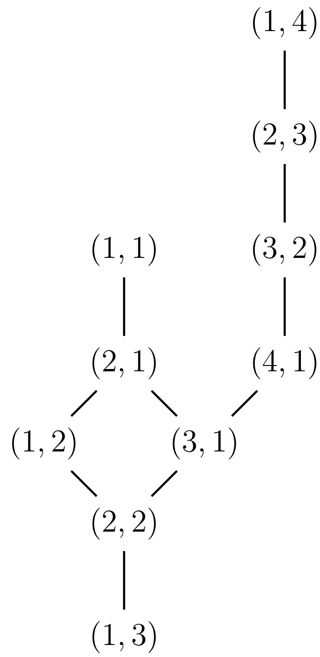
$$x = \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \end{array} \in \mathcal{B}(4)$$

which gives the following directed acyclic graph:



We then label the vertices according to their positions and construct the Hasse diagram for

P_x below:



We have the following natural definition:

Definition 3.1. *Let $x \in \mathcal{B}(n)$. An interlacing triangle of rank n, call it a, is* oriented with respect to x *if for all positions (i, j) we have the following:*

1. *if $x(i, j) = 0$ then $a(i-1, j) < a(i, j) < a(i-1, j+1)$*
2. *if $x(i, j) = 1$ then $a(i-1, j+1) < a(i, j) < a(i-1, j)$.*

Furthermore, let $\mathscr{T}_n(x)$ be the set of all interlacing triangles of rank n oriented with respect to x.

Proposition 3.2. *For all $x \in \mathcal{B}(n)$ there exists a bijection:*

$$\phi_x : \mathscr{L}(P_x) \longrightarrow \mathscr{T}_n(x).$$

Moreover, $\{\mathscr{T}_n(x)\}_{x \in \mathcal{B}(n)}$ is a set partition of $\mathscr{T}_n$.

Proof. Let σ be a linear extension of P_x for a fixed $x \in \mathcal{B}(n)$. Since σ preserves the ordering on P_x, then it induces a graph labelling on G_x such that any traversal on G_x is increasing with respect to the labelling. We can then omit the arrows to yield an interlacing triangle of rank n which is oriented with respect to x. Now let $a \in \mathscr{T}_n(x)$. We add arrows to a which connect diagonally adjacent positions, pointing in the direction of increasing value. This constitutes a labelling on G_x and by construction is a linear extension on P_x.

To show the second statement, we first prove that $\mathscr{T}_n(x) \cap \mathscr{T}_n(y) = \emptyset$ for all $x \neq y \in \mathcal{B}(n)$. Suppose $a \in \mathscr{T}_n(x) \cap \mathscr{T}_n(y)$ for $x \neq y \in \mathcal{B}(n)$. Since $x \neq y$ there exists a position (i, j) such that $x(i, j) = 0$ and $y(i, j) = 1$. This implies $a(i, j) < a(i-1, j)$ since $a \in \mathscr{T}_n(x)$ and $a(i, j) > a(i-1, j)$ since $a \in \mathscr{T}_n(y)$ which is a contradiction. We now show, $\bigcup_{x \in \mathcal{B}(n)} \mathscr{T}_n(x) = \mathscr{T}_n$.

To see this, let $a \in \mathscr{T}_n$ and connect the diagonally adjacent entries in a with edges that point in the direction of increasing value. Now ignore the labelling of this directed graph and observe that it is acyclic. Note that $a \in \mathscr{T}_n$ exhibits an orientation $x \in \mathcal{B}(n)$ defined as follows:

1. Let $x(i,j) = 0$ if $(i,j) \longrightarrow (i-1, j+1)$ and $(i,j) \longleftarrow (i-1,j)$,
2. Let $x(i,j) = 1$ if $(i,j) \longrightarrow (i-1,j)$ and $(i,j) \longleftarrow (i-1,j+1)$.

Therefore $a \in \mathscr{T}_n(x)$ for some $x \in \mathcal{B}(n)$. Finally, let $a \in \mathscr{T}_n(x)$ for some $x \in \mathcal{B}(n)$. This says that a is an interlacing triangle of rank n oriented with respect to x and therefore $a \in \mathscr{T}_n$. □

We are now ready to state the main result of this section.

Theorem 3.1. *For all $n \in \mathbb{N} \setminus \{1\}$ we have,*

$$t_n = \sum_{x \in \mathcal{B}(n)} \ell(P_x)$$

Proof. This follows immediately from Proposition 3.2. □

References

[1] G. Brightwell, P. Winkler. Counting Linear Extensions is #P-Complete, Association for Computing Machinery, New York, NY, USA *Proceedings of the Twenty-Third Annual ACM Symposium on Theory of Computing*, 175–181, 1991

[2] S. Hopkins. Order polynomial product formulas and poset dynamics, arXiv:2006.01568, 2020

[3] R. Stanley. *Enumerative Combinatorics: Volume 1*, Cambridge University Press, United States, 2011