MEDIAN ABSOLUTE DEVIATION

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For an integer $n \geq 1$ consider the set

$$A_n = \{2k^2 \ : \ k = 1, 2, \ldots, n\},$$

and let a_n denote the median absolute deviation of A_n, i.e.,

$$a_n = \text{median}(\{|x - \text{median}(A_n)| \ : \ x \in A_n\}).$$

In the comments to sequence A345318 in the OEIS [1] it is conjectured that $\lim_{n\to\infty} a_n/n^2 = \sqrt{3}/4$. The purpose of this note is to show that the conjecture holds true.

First, notice that the purpose of the factor 2 in $2k^2$ in the definition of A_n is merely to ensure that $(a_n)_{n\geq 1}$ is an integer sequence. Therefore, it has no significance in the analysis of the sequence and we shall omit it. Thus, we replace A_n by

$$A_n = \{k^2 \ : \ k = 1, 2, \ldots, n\},$$

and study $(a_n)_{n\geq 1}$ with respect to this definition.

Theorem 1. *We have* $\lim_{n\to\infty} a_n/n^2 = \sqrt{3}/8$.

Proof. First, assume that n is odd. We shall make use of the fact that for every integer $m \geq 1$, we have

$$\sum_{k=0}^{m-1} (2k - 1) = m^2.$$

Obviously, $\text{median}(A_n) = (n + 1)^2/4$. Setting

$$B_n = \{|x - \text{median}(A_n)| \ : \ x \in A_n\},$$

we have

$$\begin{aligned} B_n &= \left\{ \left| \frac{(n+1)^2}{4} \right| - k^2 \ : \ k = 1, 2, \ldots, n \right\} \\ &= \left\{ \frac{(n+1)^2}{4} - k^2 : \ k = 1, \ldots, \frac{n+1}{2} \right\} \bigcup \left\{ k^2 - \frac{(n+1)^2}{4} \ : \ k = \frac{n+1}{2} + 1, \ldots, n \right\} \end{aligned}$$

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$$= \left\{ \sum_{j=k}^{\frac{n+1}{2}-1} (2j+1) : \ k = 1, \dots, \frac{n+1}{2} \right\} \bigcup \left\{ \sum_{j=\frac{n+1}{2}}^{k-1} (2j+1) : \ k = \frac{n+1}{2} + 1, \dots, n \right\}$$

$$= \left\{ (k-1)n - (k-2)(k-1) : \ k = 1, \dots, \frac{n+1}{2} \right\} \bigcup \left\{ kn + k(k+1) : \ k = 1, \dots, \frac{n-1}{2} \right\}.$$

Let us denote by L_n (resp. R_n) the set on the left (resp. right) of the last union symbol. Clearly, $kn + k(k+1)$ is monotone increasing with k and it is easy to see that $(k-1)n - (k-2)(k-1)$ is monotone increasing with k, for $k = 1, 2, \dots, (n+1)/2$. We distinguish between two cases:

(1) $a_n \in L_n$: In this case there are $k_L \in \{1, 2, \dots, (n+1)/2\}$ and $k_R \in \{1, 2, \dots, (n-1)/2\}$ such that

$$k_L - 1 + k_R = \frac{n-1}{2}, \tag{1}$$

$$(k_L - 1)n - (k_L - 2)(k_L - 1) \geq k_R n + k_R(k_R + 1), \tag{2}$$

$$(k_L - 1)n - (k_L - 2)(k_L - 1) < (k_R + 1)n + (k_R + 1)(k_R + 2), \tag{3}$$

Substituting (1) into (2) and (3) we obtain

$$-2k_L^2 + (3n+5)k_L - \frac{3n^2 + 10n + 11}{4} \geq 0,$$

$$-2k_L^2 + (3n+7)k_L - \frac{3n^2 + 18n + 23}{4} < 0.$$

Solving these two inequalities we conclude that

$$k_L \in \left(\frac{3n + 5 - \sqrt{3n^2 + 10n + 3}}{4}, \frac{3n + 7 - \sqrt{3}(n+1)}{4} \right].$$

Since $a_n = (k_L - 1)n - (k_L - 2)(k_L - 1)$, we conclude that

$$\frac{(n-1)\sqrt{3n^2 + 10n + 3}}{8} \leq a_n \leq \frac{(n+1)^2\sqrt{3}}{8}. \tag{4}$$

Dividing (4) by n^2 and letting n go to infinity, the assertion follows.

(2) $a_n \in R_n$: In this case there are $k_L \in \{1, 2, \dots, (n+1)/2\}$ and $k_R \in \{1, 2, \dots, (n-1)/2\}$ such that

$$k_R - 1 + k_L = \frac{n-1}{2}, \tag{5}$$

$$k_R n + k_R(k_R + 1) \geq (k_L - 1)n - (k_L - 2)(k_L - 1), \tag{6}$$

$$k_R n + k_R(k_R + 1) < k_L n - (k_L - 1)k_L, \tag{7}$$

Substituting (5) into (6) and (7) we obtain

$$2k_R^2 + (n+3)k_R - \frac{n^2+2n-3}{4} \geq 0,$$
$$2k_R^2 + (n+1)k_R - \frac{(n+1)^2}{4} < 0.$$

Solving these two inequalities we conclude that

$$k_R \in \left[\frac{-n-3+\sqrt{3n^2+10n+3}}{4}, \frac{(\sqrt{3}-1)(n+1)}{4}\right).$$

Since $a_n = k_R n + k_R(k_R+1)$, we conclude that

$$\frac{(n-1)\sqrt{3n^2+10n+3}}{8} \leq a_n \leq \frac{(n+1)^2\sqrt{3}}{8}. \tag{8}$$

Dividing (8) by n^2 and letting n go to infinity, the assertion follows.

We leave the case that n is even to the interested reader. □

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.