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Finite Topological Spaces

Abstract: In this note we give generating functions that count various statistics relavent to finite topological spaces.

There is a natural bijection (sometimes called the specialization preorder) between the set of all finite topologies that can be placed on an n-set and the set of all pre-orders on $[n]$. Moreover, the same bijection shows that the T_0 topologies are in correspondence with the set of all posets on $[n]$. The bijection avails us the opportunity to view a finite topological space as a transitive digraph. Hence, the method of deriving generating functions via the symbolic method (Cf. [1]) is highly applicable.

Let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ where a_n is the number of posets on $[n]$. A001035.

Then ...

$A(\exp(x)-1) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ where b_n is the number of preorders on $[n]$. A000798

$A(y(\exp(x)-1)) = \sum_{n\geq 0} \sum_{k=0}^{n} b_{n,k} y^k \frac{x^n}{n!}$ where $b_{n,k}$ is the number of topologies on $[n]$ that have exactly k topologically distict elements. Equivalently, $b_{n,k}$ is the number of topologies τ on $[n]$ that have exactly k basis elements in the unique minimal basis of τ. A topology is T_0 if and only if $b_{n,k} = n$. A247231.

$\log(A(x)) + 1 = \sum_{n\geq 0} c_n \frac{x^n}{n!}$ where c_n is the number of connected T_0 topologies on $[n]$. A001927

$\log(A(\exp(x)-1))+1 = \sum_{n\geq 0} c_n \frac{x^n}{n!}$ where c_n is the number of connected topologies on $[n]$. A001929

$\exp(y\log(A(x))) = A(x)^y = \sum_{n\geq 0} \sum_{k=0}^{n} d_{n,k} y^k \frac{x^n}{n!}$ where $d_{n,k}$ is the number of topologies on $[n]$ that have exactly k connected components.

$\exp(y\log(A(\exp(x)-1))) = A(\exp(x)-1)^y = \sum_{n\geq 0} \sum_{k=0}^{n} d_{n,k} y^k \frac{x^n}{n!}$ where $d_{n,k}$ is the number of T_0 topologies on $[n]$ that have exactly k connected components. A247232

A set is clopen if and only if it is a union of connected components.

$A(x)^y|_{y=2} = A(x)^2 = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ where f_n is the number of ordered pairs (τ, C) with τ a T_0 topology on $[n]$ and C a clopen set in τ. Equivalently, f_n is the number of ways to build a T_0 topology on a subset S of $[n]$ then build another T_0 topology on $[n] - S$.

$A(\exp(x)-1)^y|_{y=2} = A(\exp(x)-1)^2 = \sum_{n\geq 0} g_n \frac{x^n}{n!}$ where g_n is the number of ordered pairs (τ, C) with τ a topology on $[n]$ and C a clopen set in τ. Equivalently, g_n is the number of ways to build a topology on a subset S of $[n]$ then build another topology on $[n] - S$. A281547

$A(x)^y|_{y\to 1+y} = A(x)^{1+y} = \sum_{n\geq 0}\sum_{k=0}^{n} t_{n,k} y^k \frac{x^n}{n!}$ where $t_{n,k}$ is the number of ordered pairs (τ, C) with τ a T_0 topology on $[n]$ and C a clopen set in τ with exactly k components.

$A(\exp(x)-1)^y|_{y\to 1+y} = A(\exp(x)-1)^{1+y} = \sum_{n\geq 0}\sum_{k=0}^{n} t_{n,k} y^k \frac{x^n}{n!}$ where $t_{n,k}$ is the number of ordered pairs (τ, C) with τ a topology on $[n]$ and C a clopen set in τ with exactly k components.

$(\log(A(x)+1)A(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ where f_n is the number of ordered pairs (τ, C) with τ a T_0 topology on $[n]$ and C a clopen and connected set in τ.

$(\log(A(\exp(x)-1)+1)A(\exp(x)-1) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ where f_n is the number of ordered pairs (τ, C) with τ a topology on $[n]$ and C a clopen and connected set in τ. A284762

A transitive digraph is a partial ordering of two types of atoms (reflexive and irreflexive elements) and equivalence classes of size two or more.

$A(x + x + \exp(x) - 1 - x) = A(x + \exp(x) - 1) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ where f_n is the number of transitive digraphs on n labeled nodes. A006905

$A(x+\exp(yx)-1) = \sum_{n\geq 0}\sum_{k=0}^{n} t_{n,k} y^k \frac{x^n}{n!}$ where $t_{n,k}$ is the number of transitive digraphs on n labeled nodes with exactly k reflexive points. We note that $t_{n,0}$ is the number of preorders because transitive and irreflexive imply antisymmetric. Also, $t(n,n)$ is the number of preorders. A245767

$A(x+\exp(x)-1)^y = \sum_{n\geq 0}\sum_{k=0}^{n} t_{n,k} y^k \frac{x^n}{n!}$ where $t_{n,k}$ is the number of transitive digraphs on n labeled nodes with exactly k connected components. A343882

Bibliography

[1] Phillipe Flajolet and Robert Sedgewick, Analytic Combinatorics, Cambridge Univ. Press, 2009.