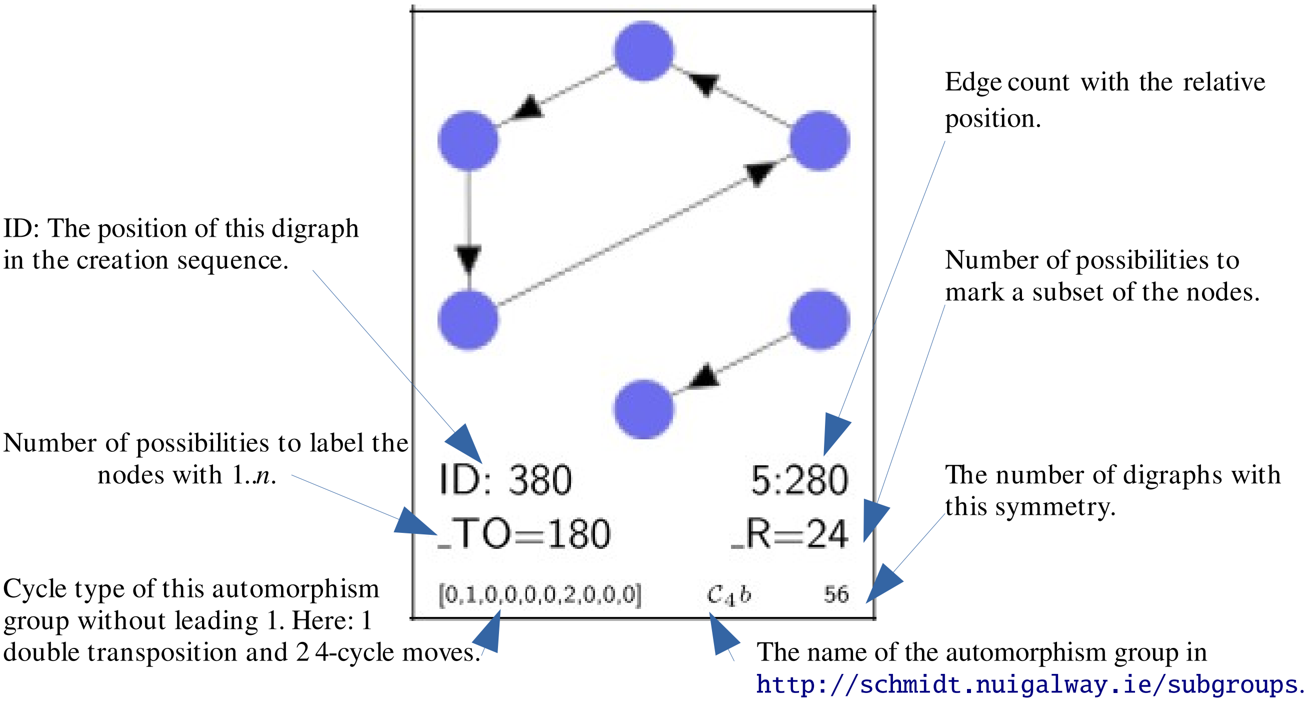
How to read the ‘Digraph Symmetry Tables’

A Digraph Symmetry Table lists digraphs of the different symmetries to a given count of n nodes. The symmetry type of a digraph is determined by its automorphism group. It is a permutation group on the nodes set, and therefore a subgroup of the symmetric group $\mathcal{S}_n$. The total number of these is determined by https://oeis.org/A000638. But not all of them occur as an automorphism group of a digraph.

The Digraph Symmetry Tables presented here bases on an enumeration algorithm sorting out digraphs with a symmetry already found. Therefore the shown digraphs are each the first one of its symmetry type found by the algorithm. Since it lists the digraphs with increasing edge count, those minimality is guaranteed for each symmetry type.

Example:



The cumulative cycle type of the permutation group indicates the quantity of member permutations, whose cycle type yields a specific partition of n. The partitions are listed in graded lexicographical ordering (see https://oeis.org/A193073) omitting the first one (1^n) representing the identity. For $n = 6$ there are 10 non-trivial partitions with $(2,1^4)$, $(2^2,1^2)$, (2^3), $(3,1^3)$, $(3,2,1)$, (3^2), $(4,1^2)$, $(4,2)$, $(5,1)$, (6), see also https://oeis.org/A000041.

The cycle type may be used to calculate different quantities. Especially the order of the permutation group is just the sum s of its values. $n!/s$ yields the index of the group in $\mathcal{S}_n$ and therefore the number of possibilities to label the nodes with the numbers $1,\ldots,n$, here designated with _TO. Multiplied with the number of digraphs with this symmetry we have the contribution of this symmetry type to the number of the labeled digraphs, which is in total $2^{(n^2-n)}$, see also https://oeis.org/A053763.

It is implicitely clear, that the sum over all digraph quantities of a symmetry table results in the total of digraphs with n nodes represented by https://oeis.org/A000273. The automorphism group of asymmetric digraphs is trivial, as it contains the identity id as the only member.

Therefore those digraph quantity accords https://oeis.org/A051504. Please note, that a digraph and its complement have the same symmetry type, so you may conclude, that an odd digraph quantity implies at least one self complementary digraph of this symmetry type.

The calculation of the number of the possibilities to mark a subset of nodes, here designated with _R, bases on the Pólya enumeration theorem. Each permutation α is assigned to the number $z(\alpha)$ of cycles in α, whereby fixpoints are cycles for its own. If α has the cycle type $(2^2, 1^2)$, so $z(\alpha) = 4$. With Pólya we get

$$\_\mathrm{R}(G) = \frac{1}{|\mathrm{Aut}(G)|} \sum_{\alpha \in \mathrm{Aut}(G)} 2^{z(\alpha)}$$

for a digraph G with automorphism group Aut(G). Using the cumulative cycle type of Aut(G) with order s and the assignment q to the partitions of n, we may formulate alternatively:

$$\_\mathrm{R}(G) = \frac{1}{s} \sum_{p \in \mathrm{Partitions}(n)} q(p) 2^{|p|}$$

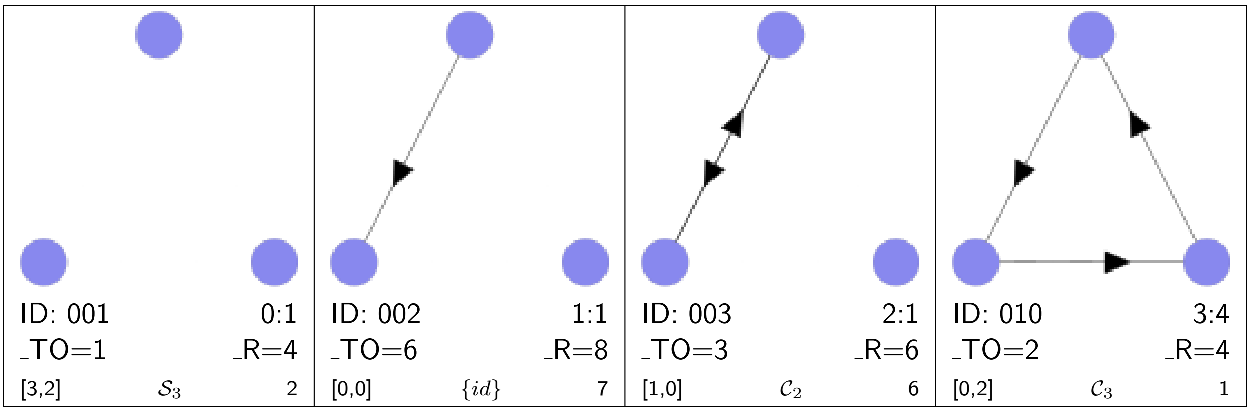
If we understand the marking of a subset of the nodes as a reflexive extension of the digraph yielding an arbitrary binary relation, we see, that the total of the _R values multiplied with the digraph quantities results in the values of https://oeis.org/A000595.

http://schmidt.nuigalway.ie/subgroups/alt.html lists all the subgroups of $\mathcal{S}_n$ for $n = 5, ..., 12$. The group identifications referring [ATLAS] are taken from there with the exception that C_n is used instead of simply „n" for the cyclic group with n elements. Note, that different permutation groups may have the same group structure, i.e. may be isomorph. E.g. $\mathcal{S}_6$ contains 3 different subgroups of the isomorphism class C_2: C_2a, C_2b, and C_2c, all with different cycle types. But however, different permutation groups may have the same cycle type: In $\mathcal{S}_6$ the subgroups C_2^2a and C_2^2b have the same cycle type [0,3,0,...], but they both do not occur as automorphism groups of a digraph with six nodes. And in $\mathcal{S}_6$ all other subgroups have different cycle types. So the cycle type identifies the automorphism group for $n \leq 6$. But there is no such statement for $n > 6$.

By the way: The number of the distinct orders of the subgroups of $\mathcal{S}_n$ is given by https://oeis.org/A218913. The number of the group isomorphism classes of the subgroups of $\mathcal{S}_n$ is given by https://oeis.org/A174511.

References

[ATLAS]	John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker, and Robert A. Wilson: Atlas of finite groups, Oxford University Press, 1985.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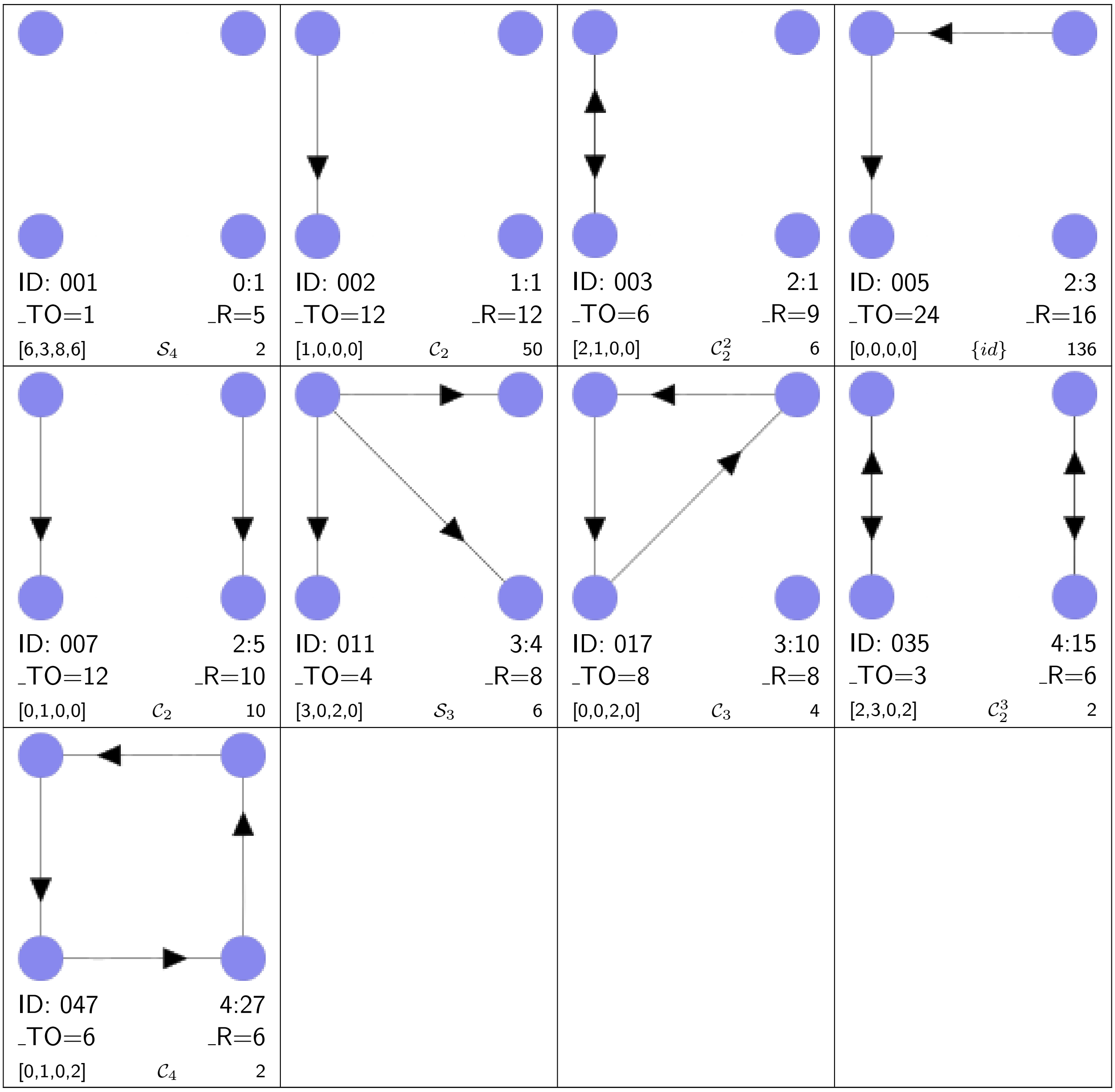
0:1
_TO=1
_R=4
[3,2]
$\mathcal{S}_3$
2
ID: 002
1:1
_TO=6
_R=8
[0,0]
$\{id\}$
7
ID: 003
2:1
_TO=3
_R=6
[1,0]
$\mathcal{C}_2$
6
ID: 010
3:4
_TO=2
_R=4
[0,2]
$\mathcal{C}_3$
1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_TO=1 _R=5
[6,3,8,6] $\mathcal{S}_4$ 2
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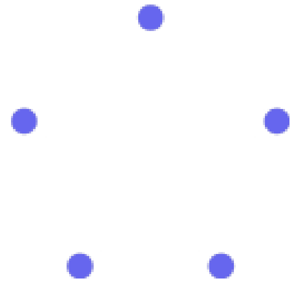
_TO=12 _R=12
[1,0,0,0] $\mathcal{C}_2$ 50
ID: 003 2:1
_TO=6 _R=9
[2,1,0,0] $\mathcal{C}_2^2$ 6
ID: 005 2:3
_TO=24 _R=16
[0,0,0,0] $\{id\}$ 136
ID: 007 2:5
_TO=12 _R=10
[0,1,0,0] $\mathcal{C}_2$ 10
ID: 011 3:4
_TO=4 _R=8
[3,0,2,0] $\mathcal{S}_3$ 6
ID: 017 3:10
_TO=8 _R=8
[0,0,2,0] $\mathcal{C}_3$ 4
ID: 035 4:15
_TO=3 _R=6
[2,3,0,2] $\mathcal{C}_2^3$ 2
ID: 047 4:27
_TO=6 _R=6
[0,1,0,2] $\mathcal{C}_4$ 2



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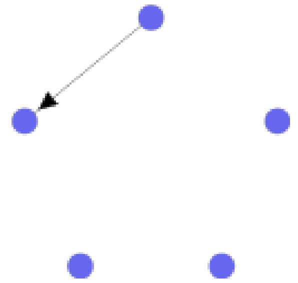
[10,15,20,20,30,24] $\mathcal{S}_5$ 2



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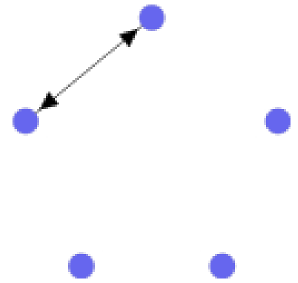
[3,0,2,0,0,0] $\mathcal{S}_3 a$ 50



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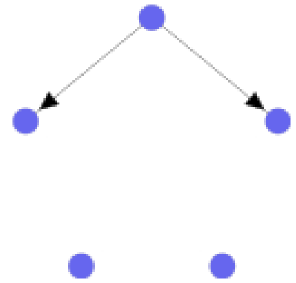
_TO=10 _R=12
[4,3,2,2,0,0] $\mathcal{D}_{12}$ 14



ID: 004 2:2
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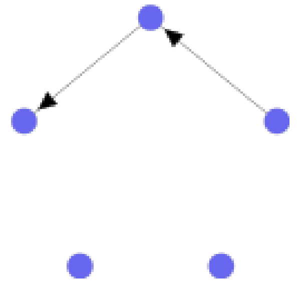
[2,1,0,0,0,0] $\mathcal{C}_2^2 b$ 106



ID: 005 2:3
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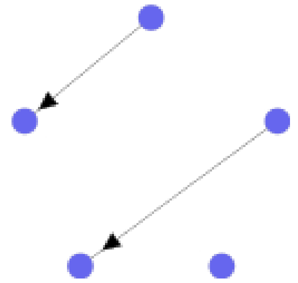
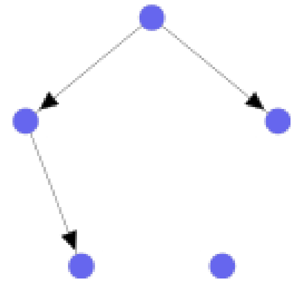
[1,0,0,0,0,0] $\mathcal{C}_2 a$ 1190



ID: 007 2:5
_TO=60 _R=20
[0,1,0,0,0,0] $\mathcal{C}_2 b$ 183



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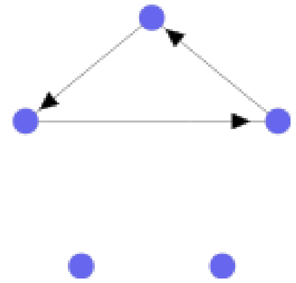
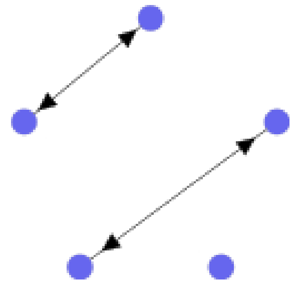
_TO=120 _R=32
[0,0,0,0,0,0] $\{id\}$ 8001



ID: 018 3:11
_TO=20 _R=12
[1,0,2,2,0,0] $\mathcal{S}_3 b$ 8



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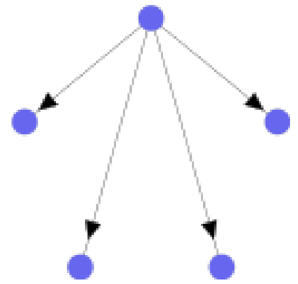
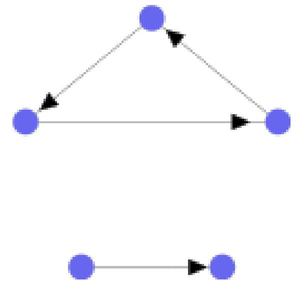
[2,3,0,0,2,0] $\mathcal{D}_8$ 8



ID: 044 4:21
_TO=5 _R=10
[6,3,8,0,6,0] $\mathcal{S}_4$ 6



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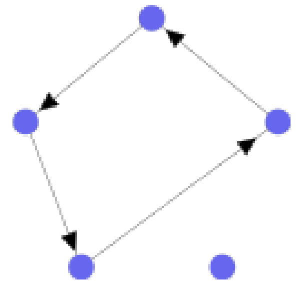
[0,0,2,0,0,0] $\mathcal{C}_3$ 28



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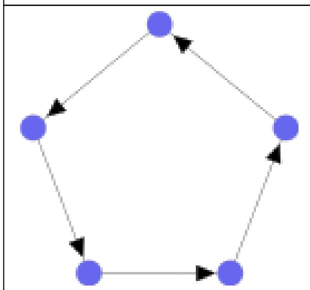
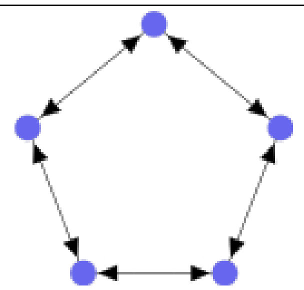
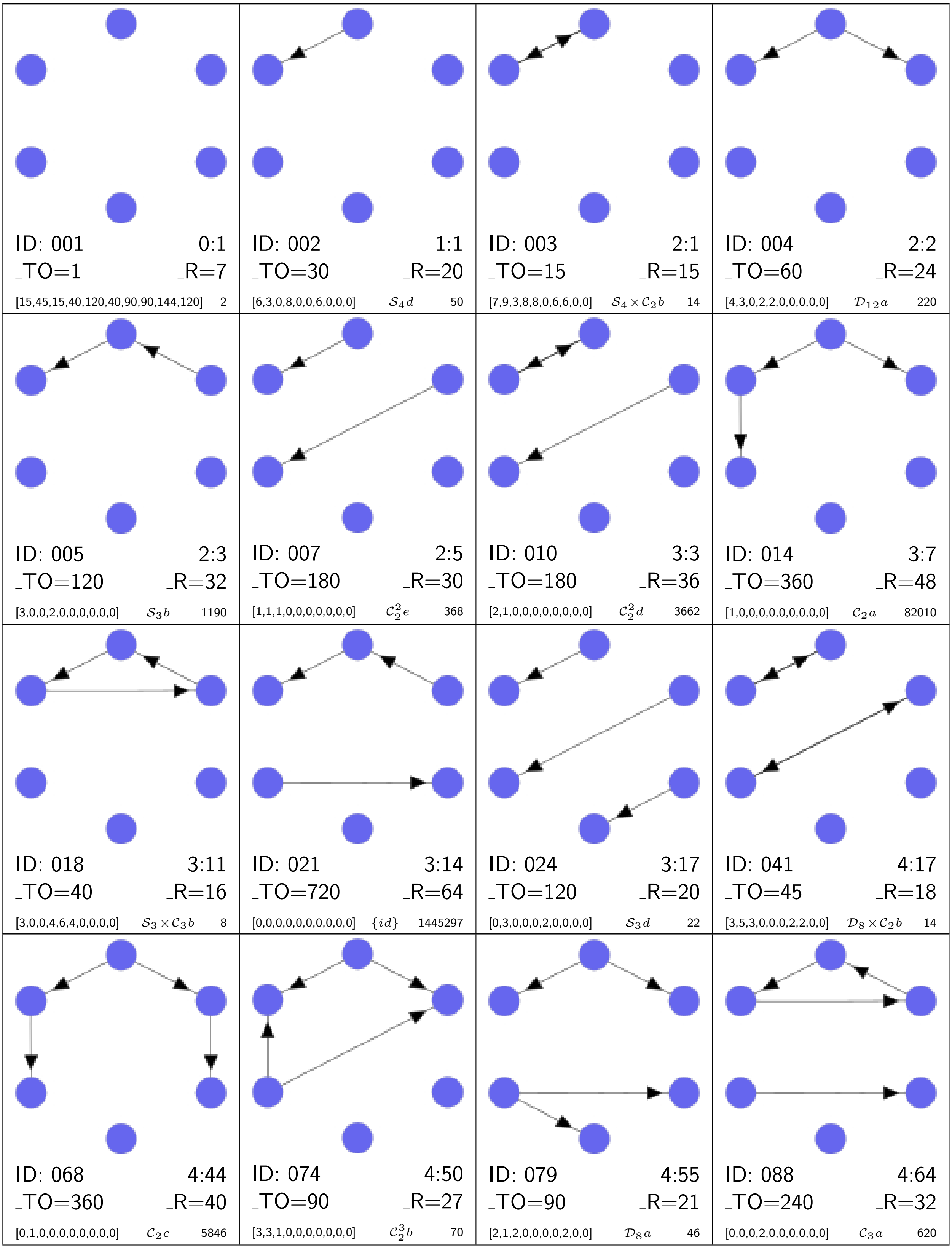
[0,1,0,0,2,0] $\mathcal{C}_4$ 8



ID: 238 5:154
_TO=24 _R=8
[0,0,0,0,0,4] $\mathcal{C}_5$ 3



ID: 5164 10:1195
_TO=12 _R=8
[0,5,0,0,0,4] $\mathcal{D}_{10}$ 1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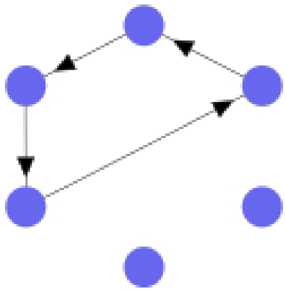
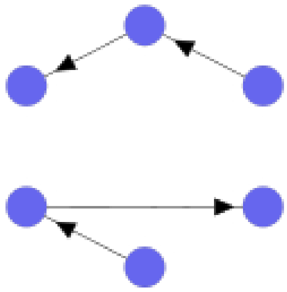
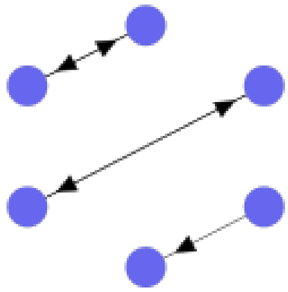
_TO=1 _R=7
[15,45,15,40,120,40,90,90,144,120] 2
ID: 002 1:1
_TO=30 _R=20
[6,3,0,8,0,0,6,0,0,0] $\mathcal{S}_4 d$ 50
ID: 003 2:1
_TO=15 _R=15
[7,9,3,8,8,0,6,6,0,0] $\mathcal{S}_4 \times \mathcal{C}_2 b$ 14
ID: 004 2:2
_TO=60 _R=24
[4,3,0,2,2,0,0,0,0,0] $\mathcal{D}_{12} a$ 220
ID: 005 2:3
_TO=120 _R=32
[3,0,0,2,0,0,0,0,0,0] $\mathcal{S}_3 b$ 1190
ID: 007 2:5
_TO=180 _R=30
[1,1,1,0,0,0,0,0,0,0] $\mathcal{C}_2^2 e$ 368
ID: 010 3:3
_TO=180 _R=36
[2,1,0,0,0,0,0,0,0,0] $\mathcal{C}_2^2 d$ 3662
ID: 014 3:7
_TO=360 _R=48
[1,0,0,0,0,0,0,0,0,0] $\mathcal{C}_2 a$ 82010
ID: 018 3:11
_TO=40 _R=16
[3,0,0,4,6,4,0,0,0,0] $\mathcal{S}_3 \times \mathcal{C}_3 b$ 8
ID: 021 3:14
_TO=720 _R=64
[0,0,0,0,0,0,0,0,0,0] $\{id\}$ 1445297
ID: 024 3:17
_TO=120 _R=20
[0,3,0,0,0,2,0,0,0,0] $\mathcal{S}_3 d$ 22
ID: 041 4:17
_TO=45 _R=18
[3,5,3,0,0,0,2,2,0,0] $\mathcal{D}_8 \times \mathcal{C}_2 b$ 14
ID: 068 4:44
_TO=360 _R=40
[0,1,0,0,0,0,0,0,0,0] $\mathcal{C}_2 c$ 5846
ID: 074 4:50
_TO=90 _R=27
[3,3,1,0,0,0,0,0,0,0] $\mathcal{C}_2^3 b$ 70
ID: 079 4:55
_TO=90 _R=21
[2,1,2,0,0,0,0,2,0,0] $\mathcal{D}_8 a$ 46
ID: 088 4:64
_TO=240 _R=32
[0,0,0,2,0,0,0,0,0,0] $\mathcal{C}_3 a$ 620



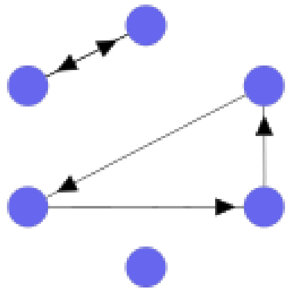
ID: 089 4:65
_TO=90 _R=18
[1,1,1,0,0,0,2,2,0,0] $\mathcal{C}_4 \times \mathcal{C}_2$ 16



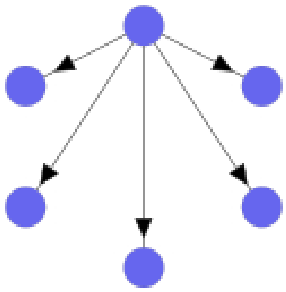
ID: 096 4:72
_TO=360 _R=36
[0,0,1,0,0,0,0,0,0,0] $\mathcal{C}_2 b$ 1100



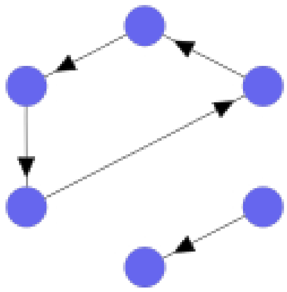
ID: 191 5:91
_TO=90 _R=24
[2,3,0,0,0,0,2,0,0,0] $\mathcal{D}_8 b$ 56



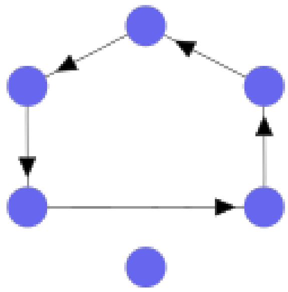
ID: 198 5:98
_TO=120 _R=24
[1,0,0,2,2,0,0,0,0,0] $\mathcal{C}_6 b$ 120



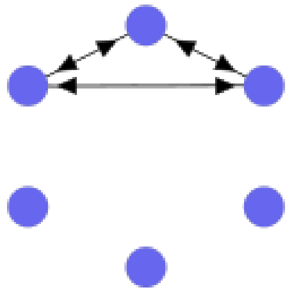
ID: 202 5:102
_TO=6 _R=12
[10,15,0,20,20,0,30,0,24,0] $\mathcal{S}_5 a$ 6



ID: 380 5:280
_TO=180 _R=24
[0,1,0,0,0,0,2,0,0,0] $\mathcal{C}_4 b$ 56



ID: 381 5:281
_TO=144 _R=16
[0,0,0,0,0,0,0,0,4,0] $\mathcal{C}_5$ 12



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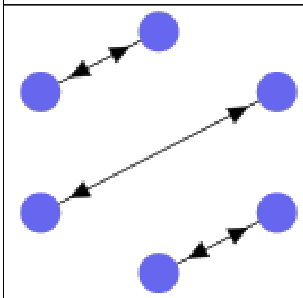
[6,9,0,4,12,4,0,0,0,0] $\mathcal{S}_3^2$ 6



ID: 880 6:492
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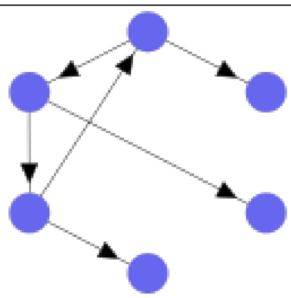
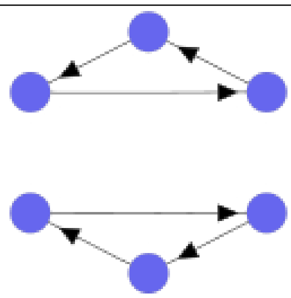
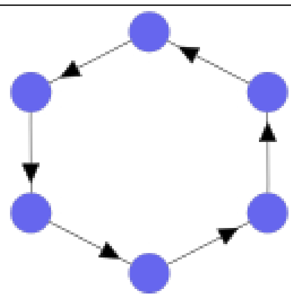


ID: 1171 6:783
_TO=240 _R=24
[0,0,0,0,0,2,0,0,0,0] $\mathcal{C}_3 b$ 62



ID: 1430 6:1042
_TO=40 _R=10
[0,0,3,4,0,4,0,0,0,6] $\mathcal{S}_3 \times \mathcal{C}_3 a$ 2



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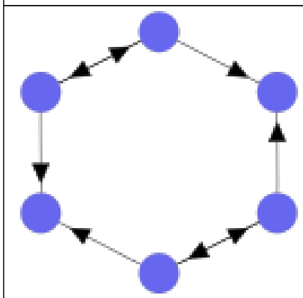

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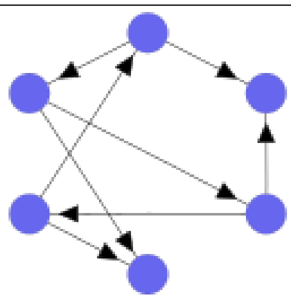
_TO=180 _R=24
[0,1,2,0,0,0,0,0,0,0] $\mathcal{C}_2^2 c$ 22



ID: 13559 8:8886
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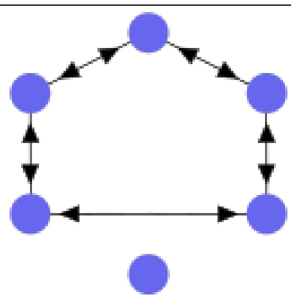
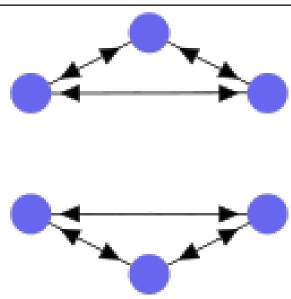


ID: 49310 10:14477
_TO=72 _R=16
[0,5,0,0,0,0,0,0,4,0] $\mathcal{D}_{10}$ 4



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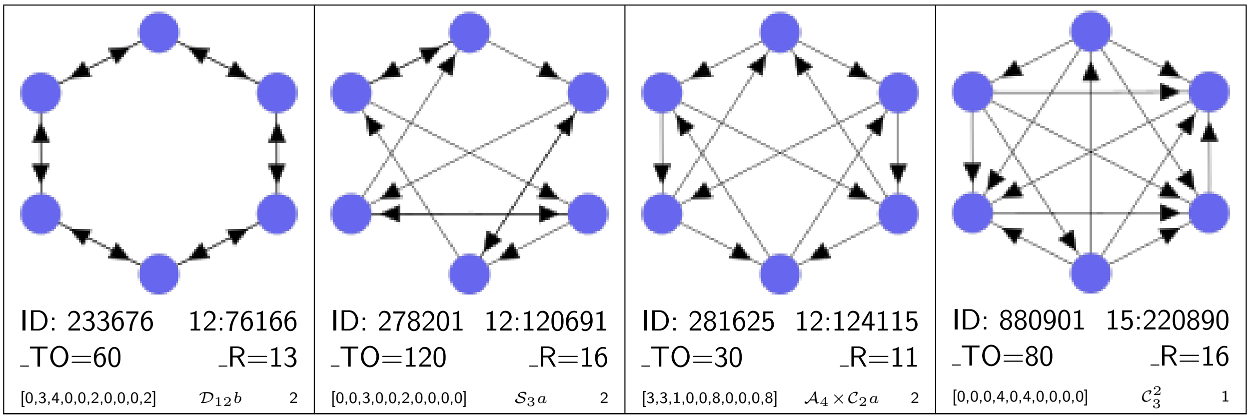
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_TO=120 _R=16
[0,0,3,0,0,2,0,0,0,0] $\mathcal{S}_3 a$ 2
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_TO=30 _R=11
[3,3,1,0,0,8,0,0,0,8] $\mathcal{A}_4 \times \mathcal{C}_2 a$ 2
ID: 880901 15:220890
_TO=80 _R=16
[0,0,0,4,0,4,0,0,0,0] $\mathcal{C}_3^2$ 1