Counting $i$-paths

Notation

$i$-path: $i$ non-intersecting paths on a lattice using only vertical and horizontal moves [i.e. monotonic non-decreasing], beginning (LHS) on $i$ points on a descending diagonal and ending (RHS) on $i$ consecutive points on a descending diagonal

$m_i = \{0, m_1, m_2, \ldots, m_{i-1}\}$: positions on $0^{\text{th}}$ descending diagonal of initial points of an $i$-path

$p(m_i, n, k)$: number of $i$-paths (of length $n \geq 0$) from $m_i$ to $i$ consecutive points on the $n^{\text{th}}$ descending diagonal beginning at column $k \geq 0$

$a(m_i, n)$: total number of $i$-paths (of length $n \geq 0$) from $m_i$ to $i$ consecutive points on the $n^{\text{th}}$ descending diagonal

$$a(m_i, n) = \sum_k p(m_i, n, k)$$

Example:

$$5 \rightarrow \cdot \quad \leftarrow 0$$
$$\cdot \quad \cdot \quad k$$

$$n \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \circ \quad \leftarrow 3$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \circ$$

$$0 \rightarrow \circ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \circ$$
$$1 \rightarrow \circ \quad \cdot \quad \cdot \quad \cdot \quad \circ$$
$$\cdot \quad \cdot \quad \cdot \quad \circ \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$m_i$ $3 \rightarrow \circ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$$4 \rightarrow \circ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Start and end points for the $4$-paths counted by $p((0,1,3,4),5,3)$

Note that for consecutive $m_i$ this is equivalent to beginning from points on the LH vertical in the same rows as the starting points as there is only one possible set of non-intersecting paths between them and those on the LH diagonal.
**1-paths** (easy, included for completeness)

\[
\begin{align*}
1 & 1 \\
1 & 5 \\
1 & 4 \ 10 \\
1 & 3 \ 6 \ 10 \\
1 & 2 \ 3 \ 4 \ 5 \\
\circ & 1 \ 1 \ 1 \ 1 \ 1 \ 1 \\
\end{align*}
\]

\[p(\{0\}, n, k)\] - the number of 1-paths from one LH point (\(\circ\)) to the point \((n \text{ up}, k \text{ across})\), \(n,k \geq 0\)

Formulae:

(Arrows show direction of movement from a general LH point to obtain a similar term of reduced index)

For paths to an individual point \((n,k)\), moving \(\circ\) up one reduces \(n\) by one, moving \(\circ\) across one reduces \(k\) by one:

\[
p(\{0\}, n, k) = p(\{0\}, n-1, k) + p(\{0\}, n, k-1)
\]

**BCs:** \(p(\{0\}, n, k) = \begin{cases} 
0 & n < 0 \text{ or } k < 0 \\
1 & n = 0 \text{ & } k = 0 
\end{cases}\)

\[
p(\{0\}, n, k) = {n+k \choose k} = {n+k \choose n}
\]

For paths to all points on the \(n\text{th}\) diagonal, moving \(\circ\) up or across one reduces \(n\) by one:

\[
a(\{0\}, n) = a(\{0\}, n-1) + a(\{0\}, n-1)
\]

**BCs:** \(a(\{0\}, n) = \begin{cases} 
0 & n < 0 \\
1 & n = 0 
\end{cases}\)

\[
a(\{0\}, n) = 2a(\{0\}, n-1)
\]

\[= \sum_{k} \binom{n}{k} = 2^n = \{1, 2, 4, 8, 16, 32, \ldots\}\]
2-paths

\[\begin{align*}
&\bullet 1 \\
&\bullet 1 \quad \bullet 15 \\
&\bullet 1 \quad \bullet 10 \quad \bullet 50 \\
&\bullet 1 \quad \bullet 6 \quad \bullet 20 \quad \bullet 50 \\
&\bullet 1 \quad \bullet 6 \quad \bullet 10 \quad \bullet 15 \\
&\circ 1 \quad \bullet 1 \quad \bullet 1 \quad \bullet 1 \quad \bullet 1 \\
&\quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ
\end{align*}\]

\[p(\{0,1\},n,k)\] - the number of 2-paths from two consecutive diagonal LH points (\(\circ\)) to the two consecutive diagonal points with top left point at \((n\text{ up, }k\text{ across})\), \(n,k \geq 0\)

Formulae:

(Arrows show direction of movement from a general LH pair of points to obtain a similar term of reduced index)

\[\begin{align*}
p(\{0,m_i\},n,k) &= p(\{0,m_i\},n-1,k) + p(\{0,m_i-1\},n,k-1) \\
&\quad + p(\{0,m_i+1\},n-1,k) + p(\{0,m_i\},n,k-1)
\end{align*}\]

BCs: \(p(\{0,m_i\},n,k)\) =

\[\begin{cases}
0 & m_i = 0 \text{ or } n < 0 \text{ or } k < 0 \\
1 & m_i = 1 \text{ and } n = 0 \text{ and } k = 0 \\
0 & m_i > 1 \text{ and } n = 0 \text{ and } k = 0
\end{cases}\]

(New?)

\[p(\{0,1\},n,k) = p(\{0,1\},n-1,k) + p(\{0,0\},n,k-1) + p(\{0,2\},n-1,k) + p(\{0,1\},n,k-1)\]

= \(p(\{0,1\},n-1,k) + p(\{0,2\},n-1,k) + p(\{0,1\},n,k-1)\)

\[\begin{align*}
a(\{0,m_i\},n) &= a(\{0,m_i\},n-1) + a(\{0,m_i-1\},n-1) \\
&\quad + a(\{0,m_i+1\},n-1) + a(\{0,m_i\},n-1)
\end{align*}\]

\[= 2a(\{0,m_i\},n-1) + a(\{0,m_i-1\},n-1) + a(\{0,m_i+1\},n-1)\]

BCs: \(a(\{0,m_i\},n)\) =

\[\begin{cases}
0 & m_i = 0 \text{ or } n < 0 \\
1 & m_i = 1 \text{ and } n = 0 \\
0 & m_i > 1 \text{ and } n = 0
\end{cases}\]

(Given by Shapiro and shown to be Catalan, although the BCs are incomplete therein.)
\( a(\{0,1\},n) = a(\{0,1\},n-1) + a(\{0,0\},n-1) + a(\{0,2\},n-1) + a(\{0,1\},n-1) \)
\[= 2a(\{0,1\},n-1) + a(\{0,2\},n-1) \]
\[= \{1, 2, 5, 14, 42, 132, \ldots\} \quad \text{(Catalan, A000108)} \]

\( a(\{0,2\},n) = a(\{0,2\},n-1) + a(\{0,1\},n-1) + a(\{0,3\},n-1) + a(\{0,2\},n-1) \)
\[= 2a(\{0,2\},n-1) + a(\{0,1\},n-1) + a(\{0,3\},n-1) \]
\[= \{0, 1, 4, 14, 48, 165, 572, \ldots\} \quad \text{(4\textsuperscript{th} convolution of Catalan, A002057)} \]

\( a(\{0,3\},n) = \{0, 0, 1, 6, 27, 110, 429, 1638, \ldots\} \quad \text{(6\textsuperscript{th} convolution of Catalan, A003517)} \]

\( a(\{0,4\},n) = \{0, 0, 0, 1, 8, 44, 208, 910, \ldots\} \quad \text{(8\textsuperscript{th} convolution of Catalan, A003518)} \]

Matrices (new?):

\[
\begin{pmatrix}
2 & 1 & 0 & \cdots & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2 & 1 & 0 \\
0 & \cdots & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
a(0,1), n-1 \\
a(0,2), n-1 \\
\vdots \\
a(0, n-1), n-1 \\
a(0, n), n-1
\end{pmatrix}
= \begin{pmatrix}
a(0,1), n-1 \\
a(0,2), n-1 \\
\vdots \\
a(0, n-1), n-1 \\
a(0, n), n-1 = 1
\end{pmatrix}
\]

For example,

\[
\begin{pmatrix}
a(\{0,1\},4) \\
a(\{0,2\},4) \\
a(\{0,3\},4) \\
a(\{0,4\},4)
\end{pmatrix}
= \begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
42 \\
48 \\
27 \\
8
\end{pmatrix}
\]

(see \( n = 4 \) terms of the Catalan and Catalan convolution sequences above).
3-paths

\[ p(\{0,1,2\},n,k) \] - the number of 3-paths from three consecutive diagonal LH points (\(\circ\)) to the three consecutive diagonal points with top left point at \((n \text{ up}, k \text{ across}), \ n, k \geq 0\)

Formulae:

(Arrows show direction of movement from a general LH triple of points to obtain a similar term of reduced index)

\[
p(\{0,m_1,m_2\},n,k) = p(\{0,m_1,m_2\},n-1,k) + p(\{0,m_1-1,m_2-1\},n-1,k-1) + p(\{0,m_1+1,m_2\},n,k-1) + p(\{0,m_1,m_2+1\},n,k) + p(\{0,m_1+1,m_2+1\},n,k-1)
\]

BCs: \[ p(\{0,m_1,m_2\},n,k) = \begin{cases} 
0 & m_1 = 0 \text{ or } m_2 = m_1 \text{ or } n < 0 \text{ or } k < 0 \\
1 & m_1 = 1 \text{ and } m_2 = 2 \text{ and } n = 0 \text{ and } k = 0 \\
0 & (m_1 > 1 \text{ or } m_2 > 2) \text{ and } n = 0 \text{ and } k = 0 
\end{cases} \]

(New?)

\[
p(\{0,1,2\},n,k) = p(\{0,1,2\},n-1,k) + p(\{0,0,1\},n,k-1) + p(\{0,2,2\},n-1,k) + p(\{0,1,1\},n,k-1) + p(\{0,1,3\},n-1,k) + p(\{0,0,2\},n,k-1) + p(\{0,2,3\},n-1,k) + p(\{0,1,2\},n,k-1)
\]

\[
= p(\{0,1,2\},n-1,k) + p(\{0,1,3\},n-1,k) + p(\{0,2,3\},n-1,k) + p(\{0,1,2\},n,k-1)
\]
\[ a(\{0, m_1, m_2\}, n) = a(\{0, m_1, m_2\}, n - 1) + a(\{0, m_1, m_2 - 1\}, n - 1) \]
\[ + a(\{0, m_1 + 1, m_2\}, n - 1) + a(\{0, m_1, m_2 - 1\}, n - 1) \]
\[ + a(\{0, m_1, m_2 + 1\}, n - 1) + a(\{0, m_1 - 1, m_2\}, n - 1) \]
\[ + a(\{0, m_1 + 1, m_2 + 1\}, n - 1) + a(\{0, m_1, m_2\}, n - 1) \]

BCs: \( a(\{0, m_1, m_2\}, n) = \begin{cases} 
0 & m_1 = 0 \text{ or } m_2 = m_1 \text{ or } n < 0 \\
1 & m_1 = 1 \& m_2 = 2 \& n = 0 \\
0 & (m_1 > 1 \text{ or } m_2 > 2) \& n = 0 
\end{cases} \)

(New? Chung et al give an alternative? system derived from a tree interpretation.)

\[ a(\{0, 1, 2\}, n) = a(\{0, 1, 2\}, n - 1) + a(\{0, 0, 1\}, n - 1) + a(\{0, 2, 2\}, n - 1) + a(\{0, 1, 1\}, n - 1) \]
\[ + a(\{0, 1, 3\}, n - 1) + a(\{0, 0, 2\}, n - 1) + a(\{0, 2, 3\}, n - 1) + a(\{0, 1, 2\}, n - 1) \]
\[ = 2a(\{0, 1, 2\}, n - 1) + a(\{0, 1, 3\}, n - 1) + a(\{0, 2, 3\}, n - 1) \]
\[ = \{1, 2, 6, 22, 92, 422, \ldots\} \quad \text{(Baxter, A001181)} \]

(The correspondence between 3-paths and Baxter numbers was established by Dulucq & Guibert by use of two bijections.)

\[ a(\{0, 1, 3\}, n) = a(\{0, 2, 3\}, n) = \{0, 1, 5, 24, 119, 615, 3303, \ldots\} \quad \text{(Not in Sloane)} \]

(Baxter convolution?)

Matrices:

\[
\begin{pmatrix}
   a(\{0, 1, 2\}, n) & a(\{0, 1, 3\}, n) & \cdots & a(\{0, 1, n-1\}, n) & a(\{0, 1, n\}, n) \\
   0 & a(\{0, 2, 3\}, n) & \cdots & a(\{0, 2, n-1\}, n) & a(\{0, 2, n\}, n) \\
   \vdots & 0 & \cdots & \vdots & \vdots \\
   0 & a(\{0, n-2, n-1\}, n) & a(\{0, n-2, n\}, n) \\
   0 & \cdots & 0 & a(\{0, n-1, n\}, n)
\end{pmatrix}
= ...
\]

(work in progress)
4-paths

\[
\begin{array}{cccccc}
\times 1 \\
\times 1 & \times 70 \\
\times 1 & \times 35 & \times 490 \\
\times 1 & \times 15 & \times 105 & \times 490 \\
\times 1 & \times 5 & \times 15 & \times 35 & \times 70 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\circ 1 \\
\circ 1 & \circ 1 & \circ 1 & \circ 1 & \circ 1 \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

\(p(\{0,1,2,3\}, n, k)\) - the number of 4-paths from four consecutive diagonal LH points (\(\circ\)) to the four consecutive diagonal points with top left point at \((n \text{ up}, k \text{ across}), \ n, k \geq 0\)

Recurrences for \(p\) and \(a\) each have 16 terms and appear to be new recurrence systems for these triangles and sequences.

\[a(\{0,1,2,3\}, n) = \{1, 2, 7, 32, 177, 1122, \ldots\}\] (Hoggatt, A005362)
i-paths

Recurrences for $p$ and $a$ each have $2^i$ terms.

Consider the rotated $p(\{0,1,2,...,i-1\}, n, k)$ triangles with terms $\binom{n}{k}$, $n,k \geq 0$:

\[
\begin{array}{c|c}
1 & 1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\quad
\begin{array}{c|c}
1 & 1 \\
1 & 1 \\
1 & 4 & 1 \\
1 & 10 & 10 & 1 \\
1 & 20 & 50 & 20 & 1 \\
1 & 35 & 175 & 175 & 35 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
\binom{n}{k}_1 & \binom{n}{k}_2 \\
1 & 1 \\
1 & 1 \\
1 & 4 & 1 \\
1 & 10 & 10 & 1 \\
1 & 20 & 50 & 20 & 1 \\
1 & 35 & 175 & 175 & 35 & 1 \\
\end{array}
\quad
\begin{array}{c|c}
\binom{n}{k}_3 & \binom{n}{k}_4 \\
1 & 1 \\
1 & 5 & 1 \\
1 & 15 & 15 & 1 \\
1 & 35 & 105 & 35 & 1 \\
1 & 70 & 490 & 490 & 70 & 1 \\
\end{array}
\]

NB: $p(\{0,1,2,...,i-1\}, n, k) = \binom{n+k}{k}$.

These appear in Fielder & Alford as triangles derived from successive columns of Pascal’s triangles, connection with i-paths is not given.

Gessel and Viennot (see also Benjamin & Cameron) give

\[
\binom{n}{k}_i = 
\begin{bmatrix}
\binom{n}{k} & \binom{n}{k+1} & \cdots \\
\binom{n+1}{k} & \binom{n+1}{k+1} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}_{i \times i}
\]

By inspection (and see Fielder & Alford),
\[
\binom{n}{k} = \prod_{j=1}^{k} \frac{(n+i-j)}{i} = \frac{n \times (n-1) \times \ldots \times (n-k+1)}{k \times (k-1) \times \ldots \times 1} = \frac{n(n-1) \ldots (n-k-1)}{1 \times 1 \ldots 1}.
\]

which generalizes the binomial formula as follows

\[
\binom{n}{k} = \binom{n-1}{i} \binom{n-2}{i} \binom{n-3}{i} \binom{n-4}{i} \ldots \binom{1}{i} = \frac{n(n-1) \ldots (n-k-1)}{k \times (k-1) \times \ldots \times 1}.
\]

(New?) As a short hand, \( \binom{n}{k} = \frac{i(n)}{k(n)} \) where \( k(n) \) is the product of terms “n: k left, i up” and \( 0(n) = 1 \). For example,

\[
\begin{align*}
\binom{5}{4}_3 &= \frac{4(5)_3}{4(3)_3} = \begin{pmatrix} 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 35 \\
\binom{5}{2}_4 &= \frac{4(5)_4}{2(2)_4} = \begin{pmatrix} 7 & 8 \\ 6 & 7 \\ 5 & 6 \\ 4 & 5 \end{pmatrix} = 490.
\end{align*}
\]
Row sums:

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & \ldots & n & (0\text{-paths}) \\
1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 2^n \\
1 & 2 & 5 & 14 & 42 & 132 & 429 & 1430 & \text{Catalan A000108} \\
1 & 2 & 6 & 22 & 92 & 422 & 2074 & 10754 & \text{Baxter A001181} \\
1 & 2 & 7 & 32 & 177 & 132 & 7898 & 60398 & \text{Hoggatt A005362} \\
1 & 2 & 8 & 44 & 310 & 2606 & 25202 & 272582 & \text{Hoggatt A005363} \\
1 & 2 & 9 & 58 & 506 & 546 & 270226 & 1038578 & \text{Hoggatt A005364} \\
\end{array}
\]

\[a([0,1,2,\ldots,i-1],n), \, 0 \leq i \leq 6, \, 0 \leq n \leq 7\]

(New?) By inspection, \(n^{th}\) column successive differences become constant, including with 1\(^{st}\) row \(n\) and even with previous row \((i = -1)\) of \([1,2,2,\ldots]\).

Convolution formulae:

<table>
<thead>
<tr>
<th>(i)</th>
<th>Row sum</th>
<th>Diagonal sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_n = a_{n-1} + a_{n-1}) {1,2,4,8,16,32,64,\ldots}</td>
<td>(2^n)</td>
</tr>
<tr>
<td>2</td>
<td>(a_n = a_{n-1} + \sum_{k=1}^{n-1} a_{k-1}a_{n-k} + a_{n-1}) {1,2,5,14,42,132,429,\ldots}</td>
<td>(a_n = a_{n-1} + a_{n-2}) {1,1,2,3,5,8,13,\ldots}</td>
</tr>
<tr>
<td>3</td>
<td>Convolution type formula? {1, 2, 6, 22, 92, 422, 2074,\ldots}</td>
<td>(a_n = a_{n-1} + \sum_{k=1}^{n-2} a_{k-1}a_{n-2-k} + a_{n-2}) {1, 1, 2, 4, 8, 17, 37,\ldots}</td>
</tr>
<tr>
<td>4</td>
<td>{1, 2, 7, 32, 177, 1122, 12898,\ldots}</td>
<td>?</td>
</tr>
</tbody>
</table>

For the \(i = 2\) row sum above, comparison with the earlier recurrence system gives

\[a([0,2],n-1) = \sum_{k=1}^{n-1} a([0,1],k-1)a([0,1],n-1-k)\,.
\]

Can this be interpreted from the path lattice? If so, can the interpretation be extended to \(i = 3\):

\[a([0,1,2],n) = 2a([0,1,2],n-1) + a([0,1,3],n-1) + a([0,2,3],n-1)\]

so

\[a([0,1,3],n-1) \text{ and } a([0,2,3],n-1) = \text{convolution form in terms of } a([0,1,2],\cdot)\,?
\]

(New?) Note the easy way of obtaining the diagonal sum formulae for \(i = 1, 2\) from the row sum formulae.
Related references


