Self-avoiding walk enumeration via the lace expansion: tables

Nathan Clisby[1], **Richard Liang**[2] **and Gordon Slade**[3]

[1]ARC Centre of Excellence for Mathematics and Statistics of Complex Systems, Department of Mathematics and Statistics,
The University of Melbourne, Victoria 3010, Australia
[2]Department of Statistics, University of California, Berkeley, CA 94720-3860, USA
[3]Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2

E-mail: N.Clisby@ms.unimelb.edu.au, rhliang@stat.berkeley.edu and slade@math.ubc.ca

Abstract. In the companion paper [1], we introduced a new method for the enumeration of self-avoiding walks based on the lace expansion. In this paper, we supplement [1] with an extensive set of tables of enumerations.

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This paper provides an extensive set of tables of enumerations of quantities associated with the lace expansion for self-avoiding walks. It is intended to supplement [1]. The enumerations are available in machine-readable form at `http://www.math.ubc.ca/~slade/lacecounts` .

We use the notation of [1]. In particular, $c_n(x)$ denotes the number of n-step SAWs on $\mathbb{Z}^d$ with $\omega(0) = 0$ and $\omega(n) = x$, $c_n = \sum_{x\in\mathbb{Z}^d} c_n(x)$ denotes the number of n-step SAWs which start at 0, and we define $\rho_n = \sum_{x\in\mathbb{Z}^d} |x|^2 c_n(x)$ ($|x|$ denotes Euclidean distance) so that $\rho_n c_n^{-1}$ is the mean-square displacement. Let p_n denote the number of unrooted undirected self-avoiding polygons (SAP) of length n, i.e., $p_n = \frac{1}{2n} c_{n-1}(e)$ where e denotes a neighbour of 0 in $\mathbb{Z}^d$.

According to the lace expansion, for $n \geq 1$ we have

$$c_n(x) = \sum_{y\in\mathbb{Z}^d:|y|=1} c_{n-1}(x-y) + \sum_{m=2}^{n} \sum_{y\in\mathbb{Z}^d} \pi_m(y) c_{n-m}(x-y) \tag{1}$$

with

$$\pi_m(x) = \sum_{\omega\in\mathcal{W}_m(x)} \sum_{L\in\mathcal{L}_m} \prod_{st\in L} U_{st}(\omega) \prod_{s't'\in\mathcal{C}(L)} (1 + U_{s't'}(\omega)). \tag{2}$$

We restrict the sum in (2) to laces with N edges, and introduce a minus sign to obtain a non-negative integer, to define

$$\pi_m^{(N)}(x) = \sum_{\omega\in\mathcal{W}_m(x)} \sum_{L\in\mathcal{L}_m^{(N)}} \prod_{st\in L} (-U_{st}(\omega)) \prod_{s't'\in\mathcal{C}(L)} (1 + U_{s't'}(\omega)), \tag{3}$$

so that

$$\pi_m(x) = \sum_{N=1}^{m-1} (-1)^N \pi_m^{(N)}(x). \tag{4}$$

See [1] for details and explanation of notation.

Let $\pi_m^{(N)} = \sum_x \pi_m^{(N)}(x)$ and $r_m^{(N)} = \sum_x |x|^2 \pi_m^{(N)}(x)$. Let

$$\pi_m = \sum_{N=1}^{m-1} (-1)^N \pi_m^{(N)} = \sum_x \pi_m(x), \quad r_m = \sum_{N=1}^{m-1} (-1)^N r_m^{(N)} = \sum_x |x|^2 \pi_m(x). \tag{5}$$

We classify contributions to (3) according to the total number δ of dimensions explored by the m-step walk ω. Let $\pi_{m,\delta}^{(N)}$ denote the contribution to $\pi_m^{(N)}$ due to walks which explore a total of δ dimensions, with the first step taken in the positive 1-direction, the first step out of this line taken in the positive 2-direction, the first step taken out of this plane in the positive 3-direction, and so on. Then

$$\pi_m^{(N)} = \sum_{\delta=1}^{d\wedge\frac{m}{2}} \alpha_d(\delta) \pi_{m,\delta}^{(N)} \quad \text{with} \quad \alpha_d(\delta) = \prod_{j=0}^{\delta-1} (2d - 2j). \tag{6}$$

Similarly, let $r_{m,\delta}^{(N)}$ denote the contribution to $r_m^{(N)}$ due to walks which explore a total of δ dimensions, with the first step taken in the positive 1-direction, the first step out of

this line taken in the positive 2-direction, the first step taken out of this plane in the positive 3-direction, and so on. Then

$$r_m^{(N)} = \sum_{\delta=1}^{d\wedge\frac{m}{2}} \alpha_d(\delta) r_{m,\delta}^{(N)}. \tag{7}$$

The theta graphs θ_n, and their analogues R_n weighted by the squared distance between vertices of degree 3, are given by

$$\theta_n = \frac{1}{2}\frac{1}{3!}\left(\pi_n^{(2)} - 3\pi_{n-1}^{(1)}\right), \quad R_n = \frac{1}{2}\frac{1}{3!}\left(r_n^{(2)} - 3\pi_{n-1}^{(1)}\right). \tag{8}$$

In this paper, we provide tables of the following:

- p_n, c_n, ρ_n for $n \le 24$ and $7 \le d \le 12$
- θ_n, R_n for $n \le 30$, $2 \le d \le 3$; and $n \le 24$, $4 \le d \le 12$
- $\pi_{m,\delta}^{(N)}, r_{m,\delta}^{(N)}$ for $m \le 30$, $2 \le \delta \le 3$, $N \le 27$; and $m \le 24$, $4 \le \delta \le 12$, $N \le 21$.

We exclude the $\delta = 1$ counts for $\pi_{m,\delta}^{(N)}, r_{m,\delta}^{(N)}$, but they are trivially given by

$$\pi_{m,1}^{(N)} = \delta_{m,N-1} \tag{9}$$

and

$$r_{m,1}^{(N)} = \begin{cases} \delta_{m,N-1} & N \text{ even} \\ 0 & N \text{ odd.} \end{cases} \tag{10}$$

Table 1. Enumeration results for $d = 7$

n	p_n	c_n	ρ_n
1	0	14	14
2	0	182	392
3	0	2 366	7 854
4	21	30 590	137 984
5	0	395 654	2 256 086
6	602	5 110 070	35 278 488
7	0	66 009 062	534 993 158
8	29 337	852 194 966	7 934 995 040
9	0	11 002 765 718	115 719 559 158
10	1 891 764	142 019 952 830	1 665 405 930 776
11	0	1 833 202 179 662	23 713 533 679 934
12	145 264 574	23 659 632 189 662	334 705 688 143 008
13	0	305 360 673 698 150	4 689 622 607 135 606
14	12 564 305 328	3 940 760 013 826 454	65 298 732 115 338 680
15	0	50 857 078 231 126 286	904 360 740 631 660 494
16	1 184 691 509 778	656 293 571 739 976 142	12 466 823 555 538 388 736
17	0	8 469 305 943 784 113 806	171 157 170 992 392 001 774
18	119 283 489 826 088	109 290 078 485 661 202 262	2 341 350 136 692 772 891 896
19	0	1 410 313 416 278 288 850 230	31 925 740 572 250 325 717 126
20	12 648 066 331 530 882	18 198 630 021 961 664 962 694	434 075 001 299 939 844 473 216
21	0	234 835 176 481 489 026 589 958	5 886 552 947 024 754 620 926 422
22	1 398 518 160 760 296 072	3 030 253 517 601 098 034 318 254	79 640 955 282 501 944 243 046 680
23	0	39 101 713 771 945 255 704 676 382	1 075 183 628 596 467 945 185 675 870
24	160 093 670 184 230 839 770	504 552 243 465 714 026 682 387 806	14 487 048 751 874 313 228 202 417 920

Table 2. Enumeration results for $d = 8$

n	p_n	c_n	ρ_n
1	0	16	16
2	0	240	512
3	0	3 600	11 792
4	28	53 776	238 592
5	0	803 504	4 498 096
6	952	11 994 096	81 153 088
7	0	179 054 640	1 420 587 056
8	55 972	2 672 126 256	24 328 844 416
9	0	39 878 886 896	409 769 856 112
10	4 417 168	595 065 468 048	6 812 168 390 080
11	0	8 879 592 484 240	112 060 950 159 120
12	420 153 664	132 491 660 323 472	1 827 511 016 287 104
13	0	1 976 912 303 612 080	29 587 811 143 042 224
14	45 471 168 240	29 496 313 445 323 888	476 089 688 917 355 072
15	0	440 098 575 225 868 624	7 620 118 122 574 474 960
16	5 408 525 283 784	6 566 302 628 140 689 744	121 404 225 732 190 451 072
17	0	97 969 968 518 462 054 352	1 926 417 600 262 072 933 712
18	691 377 275 065 232	1 461 698 348 385 616 122 224	30 458 974 669 977 173 746 752
19	0	21 808 373 396 863 539 892 464	480 062 651 662 263 694 327 920
20	93 546 917 402 070 328	325 374 770 476 507 076 809 584	7 544 701 834 493 050 723 315 584
21	0	4 854 505 122 280 192 400 621 168	118 268 699 993 926 793 637 600 624
22	13 253 373 143 354 759 632	72 427 352 538 238 368 039 098 256	1 849 636 192 603 403 066 509 788 864
23	0	1 080 589 134 633 584 722 212 134 224	28 865 714 102 512 524 091 300 707 024
24	1 950 533 810 621 627 016 520	16 121 895 426 864 166 076 568 532 880	449 611 130 349 651 996 141 415 327 360

Table 3. Enumeration results for $d = 9$

n	p_n	c_n	ρ_n
1	0	18	18
2	0	306	648
3	0	5 202	16 866
4	36	88 146	386 208
5	0	1 493 874	8 247 330
6	1 416	25 300 530	168 620 040
7	0	428 518 386	3 346 060 626
8	97 524	7 256 300 850	64 974 108 672
9	0	122 876 680 626	1 241 028 265 554
10	9 111 312	2 080 586 127 186	23 399 114 981 256
11	0	35 229 409 431 570	436 596 857 513 730
12	1 035 507 912	596 495 353 475 538	8 076 602 216 132 640
13	0	10 099 744 526 658 546	148 336 802 423 681 346
14	135 002 944 224	171 003 188 767 881 906	2 707 771 360 971 169 224
15	0	2 895 335 387 107 970 706	49 168 711 106 271 782 226
16	19 482 335 286 552	49 021 668 492 861 718 674	888 747 740 841 709 133 568
17	0	829 999 403 731 225 961 874	16 000 168 083 012 128 218 194
18	3 039 956 387 533 248	14 052 840 969 325 999 278 258	287 030 909 777 000 351 481 288
19	0	237 930 859 323 785 632 760 370	5 132 857 882 169 674 650 178 818
20	504 644 784 188 469 480	4 028 424 846 336 006 393 611 250	91 528 997 868 382 721 199 210 144
21	0	68 205 585 749 645 874 319 685 874	1 627 977 084 204 421 571 900 330 466
22	88 092 778 444 829 586 432	1 154 790 442 297 055 887 026 756 498	28 888 984 164 012 872 052 194 639 496
23	0	19 551 790 737 608 267 993 216 369 490	511 564 983 569 377 562 587 490 018 802
24	16 031 826 321 293 947 667 016	331 031 180 656 504 515 248 478 230 994	9 041 334 505 376 758 510 237 191 750 528

Table 4. Enumeration results for $d = 10$

n	p_n	c_n	ρ_n
1	0	20	20
2	0	380	800
3	0	7 220	23 220
4	45	136 820	593 600
5	0	2 593 100	14 161 100
6	2 010	49 121 660	323 560 080
7	0	930 556 460	7 177 055 660
8	158 715	17 625 825 740	155 806 715 360
9	0	333 857 601 020	3 327 444 000 540
10	17 138 412	6 323 384 122 580	70 153 120 964 720
11	0	119 767 717 450 100	1 463 775 262 656 020
12	2 267 554 620	2 268 399 952 520 660	30 282 354 727 619 040
13	0	42 963 566 150 826 380	622 005 249 256 384 460
14	346 442 520 060	813 721 674 662 589 980	12 698 567 901 661 234 640
15	0	15 411 746 407 417 290 020	257 893 510 070 143 193 220
16	58 942 273 234 050	291 893 918 240 586 194 660	5 213 724 354 559 832 029 280
17	0	5 528 387 235 193 561 980 740	104 983 209 707 742 483 552 740
18	10 901 754 268 455 540	104 705 682 990 258 791 936 540	2 106 474 495 688 862 079 817 680
19	0	1 983 088 865 824 542 959 434 940	42 133 217 710 462 313 221 830 620
20	2 155 404 458 487 598 074	37 558 921 405 099 250 412 242 300	840 362 612 748 215 030 699 576 480
21	0	711 351 287 961 645 290 529 969 020	16 718 723 482 643 456 483 073 711 420
22	449 991 775 833 254 058 420	13 472 695 702 762 071 088 671 214 580	331 846 718 935 120 043 432 386 737 200
23	0	255 167 246 010 321 746 624 740 564 100	6 572 940 466 986 130 589 813 778 491 300
24	98 296 087 918 995 487 512 990	4 832 756 707 052 478 587 170 415 132 180	129 941 496 091 974 268 058 781 276 951 840

Table 5. Enumeration results for $d = 11$

n	p_n	c_n	ρ_n
1	0	22	22
2	0	462	968
3	0	9 702	30 998
4	55	203 302	875 072
5	0	4 260 542	23 064 910
6	2 750	89 253 582	582 414 712
7	0	1 869 809 502	14 279 929 598
8	244 915	39 167 457 582	342 703 686 688
9	0	820 458 452 462	8 091 580 351 822
10	30 019 132	17 185 914 925 542	188 619 129 280 120
11	0	359 989 506 212 182	4 351 605 736 266 822
12	4 534 342 450	7 540 511 273 930 822	99 544 303 990 679 328
13	0	157 947 298 263 243 742	2 260 917 384 277 518 318
14	795 158 855 040	3 308 420 553 034 902 382	51 040 966 658 891 338 712
15	0	69 299 392 385 043 268 822	1 146 267 572 930 261 937 430
16	156 065 769 208 030	1 451 565 583 054 963 249 302	25 626 044 210 009 817 360 512
17	0	30 404 929 596 858 248 780 502	570 619 629 992 075 324 120 438
18	33 455 709 852 728 680	636 869 613 282 097 404 699 182	12 661 377 698 527 672 639 046 808
19	0	13 340 039 327 784 366 568 293 582	280 060 059 093 199 696 996 653 342
20	7 699 443 666 570 581 398	279 423 686 104 297 325 939 502 462	6 177 295 403 233 670 777 904 786 240
21	0	5 852 876 455 869 447 505 259 219 582	135 907 285 071 737 936 921 528 781 582
22	1 878 349 519 387 046 796 520	122 595 691 140 615 738 285 574 648 182	2 983 235 983 536 604 822 396 389 140 088
23	0	2 567 917 546 820 289 870 716 780 157 382	65 346 513 665 785 491 847 374 025 434 438
24	481 123 252 983 738 780 934 750	53 788 168 727 245 424 866 692 455 320 262	1 428 647 679 624 961 577 333 220 624 030 592

Table 6. Enumeration results for $d = 12$

n	p_n	c_n	ρ_n
1	0	24	24
2	0	552	1 152
3	0	12 696	40 344
4	66	291 480	1 246 464
5	0	6 692 424	35 971 656
6	3 652	153 614 760	994 735 968
7	0	3 526 063 752	26 713 505 928
8	362 142	80 931 227 016	702 251 083 200
9	0	1 857 565 708 968	18 163 703 002 728
10	49 678 200	42 634 594 787 160	463 846 475 799 456
11	0	978 544 945 823 832	11 723 867 668 479 384
12	8 437 633 424	22 459 264 078 075 992	293 820 544 576 104 768
13	0	515 478 463 349 872 200	7 311 457 885 965 521 736
14	1 671 196 200 168	11 831 064 537 706 447 464	180 841 846 308 702 776 160
15	0	271 542 137 952 854 806 776	4 449 720 113 409 303 632 184
16	372 042 038 187 468	6 232 321 082 672 399 260 152	108 993 273 452 638 292 559 936
17	0	143 041 632 747 658 763 159 736	2 659 134 332 538 971 281 874 424
18	90 827 540 574 992 408	3 283 028 620 369 535 296 924 392	64 647 735 597 664 979 193 907 296
19	0	75 350 633 086 861 362 709 510 440	1 566 769 676 346 801 983 181 959 016
20	23 895 521 468 809 410 948	1 729 413 736 555 694 364 257 240 424	37 864 863 639 679 836 060 428 344 896
21	0	39 692 725 190 511 323 920 669 792 488	912 781 647 453 908 162 380 218 289 512
22	6 687 637 099 636 827 385 272	911 009 259 819 999 998 697 513 672 024	21 953 294 906 105 305 927 323 477 069 600
23	0	20 909 067 930 293 907 689 071 201 132 152	526 894 960 222 433 445 901 822 422 630 840
24	1 971 495 720 215 734 080 086 620	479 895 271 559 400 621 984 606 896 091 096	12 621 694 490 266 831 388 085 194 021 914 560

Table 7. θ_n for $d = 2$

n	θ_n	R_n
7	2	2
8	0	0
9	12	12
10	6	16
11	62	62
12	60	160
13	338	402
14	430	1 144
15	1 966	2 926
16	2 794	7 728
17	11 772	20 652
18	17 898	53 384
19	71 390	141 710
20	114 496	370 000
21	438 112	971 168
22	731 698	2 544 568
23	2 718 114	6 669 314
24	4 681 116	17 451 712
25	17 013 354	45 745 114
26	30 025 926	119 694 888
27	107 283 688	313 492 200
28	193 174 670	820 764 592
29	680 889 338	2 148 443 962
30	1 246 653 988	5 626 948 128

Table 8. θ_n for $d = 3$

n	θ_n	R_n
7	18	18
8	24	48
9	344	360
10	582	1 200
11	5 934	6 798
12	12 120	26 064
13	104 250	135 162
14	239 610	542 280
15	1 877 626	2 734 938
16	4 655 982	11 211 360
17	34 476 636	55 509 996
18	90 358 962	232 053 576
19	643 362 294	1 132 615 398
20	1 760 007 240	4 810 299 312
21	12 172 430 476	23 245 891 404
22	34 458 804 366	99 924 798 216
23	233 031 911 058	479 557 133 346
24	678 443 632 404	2 081 064 930 576
25	4 506 594 576 402	9 936 794 140 530
26	13 431 562 497 642	43 454 881 651 080
27	87 920 268 255 688	206 692 622 752 392
28	267 312 031 127 190	909 677 184 607 056
29	1 728 442 241 408 178	4 313 993 735 376 738
30	5 346 094 803 342 356	19 088 210 395 449 504

Table 9. θ_n for $d = 4$

n	θ_n	R_n
7	60	60
8	96	192
9	2 104	2 168
10	5 172	10 464
11	72 324	82 692
12	209 944	434 944
13	2 526 492	3 198 108
14	8 121 588	17 478 864
15	90 401 076	125 236 532
16	312 762 684	705 203 808
17	3 312 222 312	4 972 084 104
18	12 115 611 916	28 689 935 856
19	123 909 883 524	200 029 212 708
20	473 454 234 720	1 176 778 621 536
21	4 718 365 122 576	8 140 818 087 696
22	18 674 078 481 996	48 636 717 354 192
23	182 404 365 656 892	334 597 942 757 052
24	743 166 587 747 880	2 024 191 341 541 056

Table 10. θ_n for $d = 5$

n	θ_n	R_n
7	140	140
8	240	480
9	7 200	7 360
10	20 100	40 480
11	373 940	417 140
12	1 297 120	2 651 680
13	20 482 940	25 099 260
14	80 059 660	168 574 960
15	1 166 756 300	1 549 767 820
16	4 943 915 140	10 801 509 120
17	68 606 609 160	97 720 817 640
18	308 743 195 820	701 408 577 840
19	4 143 504 945 380	6 281 401 399 940
20	19 543 338 288 400	46 157 718 733 600
21	255 976 405 419 800	410 637 632 262 680
22	1 253 788 787 087 140	3 075 141 315 024 880
23	16 120 767 040 411 020	27 237 248 400 577 580
24	81 448 537 556 649 240	207 153 879 958 081 440

Table 11. θ_n for $d = 6$

n	θ_n	R_n
7	270	270
8	480	960
9	18 340	18 660
10	54 570	109 680
11	1 276 530	1 397 490
12	4 875 060	9 904 800
13	95 777 790	114 018 750
14	421 012 050	875 479 560
15	7 613 480 210	9 750 674 130
16	36 715 788 150	78 703 234 320
17	632 505 316 740	862 411 391 700
18	3 262 243 703 430	7 223 637 537 720
19	54 427 790 622 930	78 402 707 524 770
20	295 553 720 230 560	675 921 273 583 920
21	4 820 700 778 308 080	7 293 692 996 278 320
22	27 269 713 407 492 750	64 345 105 837 160 520
23	437 405 933 569 307 310	691 859 523 520 694 670
24	2 558 062 803 739 473 060	6 219 409 618 913 370 240

Table 12. θ_n for $d = 7$

n	θ_n	R_n
7	462	462
8	840	1 680
9	39 032	39 592
10	120 666	242 256
11	3 407 922	3 680 082
12	13 811 000	27 962 480
13	325 088 358	378 258 342
14	1 541 709 750	3 181 087 224
15	33 319 513 206	41 425 883 926
16	175 308 463 554	371 053 589 088
17	3 610 823 140 932	4 749 828 208 692
18	20 469 411 884 558	44 532 915 874 104
19	408 993 205 934 250	564 863 902 220 730
20	2 453 187 673 978 776	5 484 809 352 524 880
21	48 019 259 385 438 852	69 219 200 052 017 988
22	301 065 538 312 850 178	691 016 068 809 093 048
23	5 808 065 001 656 182 734	8 697 047 850 702 909 054
24	37 737 004 379 033 336 076	88 798 936 623 482 241 072

Table 13. θ_n for $d = 8$

n	θ_n	R_n
7	728	728
8	1 344	2 688
9	73 584	74 480
10	233 352	468 160
11	7 729 064	8 261 288
12	32 606 000	65 868 544
13	894 947 480	1 022 790 552
14	4 473 576 520	9 182 283 040
15	112 494 029 256	136 561 711 048
16	629 652 156 952	1 320 886 015 680
17	15 094 193 217 936	19 284 246 630 096
18	91 641 217 107 832	196 849 374 281 056
19	2 134 371 087 140 264	2 847 883 142 538 728
20	13 777 550 026 649 920	30 294 758 084 903 872
21	315 016 437 707 249 376	436 335 421 225 858 784
22	2 132 955 234 606 291 256	4 795 448 054 660 551 456
23	48 176 347 540 221 618 648	68 937 204 740 435 636 696
24	338 892 915 663 355 532 304	777 927 684 484 349 334 656

Table 14. θ_n for $d = 9$

n	θ_n	R_n
7	1 080	1 080
8	2 016	4 032
9	127 104	128 448
10	410 472	823 104
11	15 605 064	16 548 552
12	67 762 944	136 678 464
13	2 125 423 512	2 394 505 368
14	11 041 341 048	22 578 071 904
15	316 697 717 496	377 059 633 272
16	1 855 974 717 864	3 868 040 671 488
17	50 758 564 137 552	63 315 958 539 024
18	324 517 932 134 136	690 462 910 761 696
19	8 634 094 518 896 424	11 198 425 958 397 288
20	58 951 079 161 961 760	127 997 722 374 808 896
21	1 542 685 746 235 206 576	2 067 693 511 504 265 136
22	11 086 045 373 826 741 096	24 532 644 381 786 299 232
23	287 210 272 530 149 011 512	395 841 317 782 835 081 592
24	2 149 835 979 225 379 365 360	4 841 238 337 623 101 134 656

Table 15. θ_n for $d = 10$

n	θ_n	R_n
7	1 530	1 530
8	2 880	5 760
9	205 500	207 420
10	672 750	1 348 560
11	28 876 230	30 431 430
12	128 154 540	258 195 360
13	4 522 497 930	5 036 154 570
14	24 193 341 270	49 333 092 120
15	779 520 187 350	913 353 719 190
16	4 732 845 274 530	9 814 520 006 640
17	145 417 042 454 220	177 829 648 475 580
18	967 852 649 029 890	2 044 151 206 053 480
19	28 961 814 319 905 510	36 689 660 781 603 030
20	206 662 987 104 508 800	444 331 963 607 389 200
21	6 092 692 337 322 705 600	7 946 337 488 724 668 160
22	45 903 914 529 452 501 130	100 331 839 708 251 433 560
23	1 342 394 001 842 561 179 290	1 793 418 626 721 217 065 210
24	10 561 944 550 170 085 961 580	23 429 120 011 551 640 399 680

Table 16. θ_n for $d = 11$

n	θ_n	R_n
7	2 090	2 090
8	3 960	7 920
9	315 480	318 120
10	1 043 790	2 091 760
11	49 929 110	52 352 630
12	225 391 320	453 706 000
13	8 839 140 530	9 748 993 650
14	48 389 520 850	98 456 513 320
15	1 730 332 169 170	2 000 447 375 410
16	10 804 550 394 550	22 317 456 179 040
17	368 445 232 150 380	443 124 473 639 100
18	2 532 288 251 043 210	5 317 186 865 008 040
19	84 180 354 493 237 310	104 554 337 867 950 190
20	622 400 072 185 778 920	1 327 763 172 349 638 640
21	20 413 714 004 452 628 860	26 022 231 940 963 028 540
22	159 822 064 265 168 224 150	345 891 179 343 011 390 440
23	5 208 501 720 795 801 452 970	6 779 714 130 786 318 336 890
24	42 689 269 019 480 454 516 420	93 565 554 726 381 746 105 680

Table 17. θ_n for $d = 12$

n	θ_n	R_n
7	2 772	2 772
8	5 280	10 560
9	464 552	468 072
10	1 550 076	3 105 696
11	81 767 532	85 379 052
12	374 189 640	752 716 800
13	16 145 082 228	17 664 121 332
14	90 047 117 820	182 896 318 704
15	3 538 541 182 396	4 044 798 086 076
16	22 604 139 132 564	46 542 363 049 248
17	847 117 934 652 792	1 004 487 413 097 432
18	5 976 735 069 260 868	12 490 456 290 118 224
19	218 523 417 451 594 860	266 890 211 540 762 700
20	1 663 471 221 342 058 656	3 526 235 357 007 459 360
21	60 081 927 611 163 482 992	75 118 482 586 334 199 408
22	485 554 009 136 589 619 908	1 042 469 015 740 517 350 128
23	17 451 486 259 459 164 078 804	22 222 303 849 820 875 484 244
24	147 983 007 059 842 602 858 936	321 208 054 085 469 867 277 632

Table 18. $\pi^{(1)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
4	1	0	0	0	0	0
5	0	0	0	0	0	0
6	3	4	0	0	0	0
7	0	0	0	0	0	0
8	14	62	27	0	0	0
9	0	0	0	0	0	0
10	70	970	1205	248	0	0
11	0	0	0	0	0	0
12	372	15691	44885	24105	2830	0
13	0	0	0	0	0	0
14	2058	263011	1612394	1750966	507892	38232
15	0	0	0	0	0	0
16	11752	4554640	57997702	115868170	63383154	11369212
17	0	0	0	0	0	0
18	68706	81031821	2114320602	7421606913	6898287669	2235340662
19	0	0	0	0	0	0
20	409130	1473595660	78373145110	471502348157	706062497596	370624649525
21	0	0	0	0	0	0
22	2472646	27283008423	2953668569401	30035518411635	70264146764801	56436924210755
23	0	0	0	0	0	0
24	15127620	512698051540	113028684209992	1927800922616783	6912750632303847	8212024881628664
25	0	0				
26	93504944	9755593766256				
27	0	0				
28	583032968	187612021501492				
29	0	0				
30	3662883960	3641174113503125				

Table 19. $\pi^{(1)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
16	593859	0	0	0	0
17	0	0	0	0	0
18	271232325	10401712	0	0	0
19	0	0	0	0	0
20	78919923015	6899770620	202601898	0	0
21	0	0	0	0	0
22	18861199407028	2834155627248	187006205826	4342263000	0
23	0	0	0	0	0
24	4059999467284893	936251966893207	104522857093355	5391812403300	101551822350

Table 20. $\pi^{(2)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
5	3	0	0	0	0	0
6	0	0	0	0	0	0
7	12	15	0	0	0	0
8	0	6	0	0	0	0
9	60	263	106	0	0	0
10	9	141	90	0	0	0
11	303	4347	5145	966	0	0
12	90	2985	5057	1150	0	0
13	1623	72882	200640	100095	10755	0
14	645	59580	223929	130761	15024	0
15	9123	1256965	7427881	7545109	2026654	141271
16	4191	1161900	9192360	10708230	3034026	210870
17	52914	22274250	273192681	511323717	260425410	43581285
18	26847	22576317	367321358	778339209	413987447	68757977
19	313203	403882494	10134738765	33297284604	28875212754	8765135559
20	171744	439915938	14575465398	53730206382	48270912915	14475677754
21	1884558	7463566015	381046873747	2141089231129	2991982904400	1474320780749
22	1097547	8614152318	579257739210	3627384298137	5215209035700	2527059566316
23	11495109	140104964448	14532013655745	137641166185080	300222840350196	226572371921355
24	7021674	169607397264	23139151290780	242935903148196	541798309482647	400204556480735
25	70902891	2664730038705				
26	45038889	3357868104966				
27	441440364	51246767899924				
28	289762005	66827862900795				
29	2770432911	994946114189517				
30	1869980982	1336522765845098				

Table 21. $\pi^{(2)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
17	2136990	0	0	0	0
18	3234468	0	0	0	0
19	1003202376	36572274	0	0	0
20	1588273224	54532872	0	0	0
21	297024033400	24765210124	698531550	0	0
22	487207016436	38286098142	1011388752	0	0
23	71777499319176	10309600927836	654961751316	14729974326	0
24	121039695755501	16380902914538	975753794432	20594192604	0

Table 22. $\pi^{(3)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
6	5	0	0	0	0	0
7	0	0	0	0	0	0
8	27	30	0	0	0	0
9	0	12	0	0	0	0
10	150	628	225	0	0	0
11	18	346	204	0	0	0
12	828	11321	12151	2065	0	0
13	212	8196	13162	2768	0	0
14	4690	200617	503488	229199	22641	0
15	1710	177094	633836	344708	37012	0
16	27444	3602716	19387981	18020447	4470921	290490
17	12056	3670278	27557826	29946076	7942038	520488
18	164404	65891466	733216060	1255267660	591752607	92628188
19	81642	74801904	1149282612	2268440516	1129361068	176908224
20	1000424	1225857545	27774871760	83289725066	66879176600	19026797271
21	544966	1515214218	47154180650	161431124220	135594737862	38340377486
22	6163068	23145314506	1061533187186	5426942183752	7017173449606	3243776533307
23	3609666	30649241926	1925417988368	11154385468106	14965360064484	6832473605256
24	38359768	442525495287	41025209078343	352223654547146	709900983824681	502736787715520
25	23817032	620457491228				
26	240816104	8551771203968				
27	156926600	12583884947238				
28	1522857652	166785303178540				
29	1033831178	255825849298092				
30	9690897366	3278760853928382				

Table 23. $\pi^{(3)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
18	4286520	0	0	0	0
19	7903792	0	0	0	0
20	2061159992	71691210	0	0	0
21	3996106268	131129056	0	0	0
22	619728237143	49419632919	1342166690	0	0
23	1251319823298	94026978904	2387813292	0	0
24	151226097538442	20811925817977	1275670696788	27829279324	0

Table 24. $\pi^{(4)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
7	7	0	0	0	0	0
8	0	0	0	0	0	0
9	50	49	0	0	0	0
10	0	18	0	0	0	0
11	316	1221	392	0	0	0
12	27	651	342	0	0	0
13	1923	24188	23527	3661	0	0
14	410	17691	25635	4926	0	0
15	11593	455514	1037619	434229	39858	0
16	3899	420686	1358683	676233	67692	0
17	70665	8540289	41628349	35589338	8223696	501984
18	30971	9362753	63069428	62657581	15482260	958296
19	436352	161422501	1620535331	2548234047	1119557048	164995424
20	227817	201487731	2757798702	4962921782	2299189801	339847496
21	2722979	3083738455	62726520424	172306384543	128889505076	34561897395
22	1615480	4260102100	117265454130	364683600743	284540706044	75849732467
23	17139272	59522178152	2437992334812	11377804363649	13689144111432	5966900531067
24	11221939	89221130055	4923678055178	25802229897502	32088336086408	13797084673921
25	108677845	1159647989401				
26	76989236	1859161375291				
27	693493127	22779627250226				
28	524137733	38642742688293				
29	4449705104	450725399476820				
30	3550938593	802368519116504				

Table 25. $\pi^{(4)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
19	7242208	0	0	0	0
20	14462130	0	0	0	0
21	3559043167	118442394	0	0	0
22	7526537607	236700594	0	0	0
23	1085227525756	82977718903	2172988811	0	0
24	2404761754146	173201888521	4237457388	0	0

Table 26. $\pi^{(5)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
8	9	0	0	0	0	0
9	0	0	0	0	0	0
10	81	72	0	0	0	0
11	0	24	0	0	0	0
12	586	2090	615	0	0	0
13	36	1052	504	0	0	0
14	3924	45512	40677	5886	0	0
15	680	32326	43060	7696	0	0
16	25324	914361	1909047	743145	64080	0
17	7476	839944	2489802	1153144	108984	0
18	161878	17968822	79927570	63462707	13779620	796128
19	65950	19982532	122888312	113388110	26425668	1559208
20	1035910	352159679	3208982094	4673782093	1928274648	269203008
21	522632	452764086	5621832578	9366151978	4085701274	575029868
22	6655064	6926587566	127141988533	322370405867	226157350800	57461182280
23	3914286	9973144990	247434347422	709615542880	520247807650	131886303720
24	42936796	136978921166	5032839627078	21593732715258	24323867244113	10043876063048
25	28372860	215973947684				
26	278147320	2724383852464				
27	201524776	4627820402792				
28	1808632828	54483016887017				
29	1412505434	98501892987808				
30	11800899190	1095116502650373				

Table 27. $\pi^{(5)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
20	11260053	0	0	0	0
21	23468840	0	0	0	0
22	5646222303	180255645	0	0	0
23	12542277460	379921104	0	0	0
24	1744614252005	128175070578	3240836143	0	0

Table 28. $\pi^{(6)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
9	11	0	0	0	0	0
10	0	0	0	0	0	0
11	120	99	0	0	0	0
12	0	30	0	0	0	0
13	992	3283	902	0	0	0
14	45	1549	690	0	0	0
15	7303	78357	65197	8888	0	0
16	1022	53269	66157	11150	0	0
17	50611	1681182	3251822	1190147	97317	0
18	12889	1507584	4148221	1809413	163000	0
19	340879	34698269	142177853	105836771	21767510	1198329
20	125985	38325759	217300926	188240589	41743394	2364090
21	2269210	706431396	5894062093	8020909209	3129796334	416261059
22	1078675	915089587	10398739617	16200860157	6709315746	904948883
23	15042427	14328089388	239324510338	564799724827	374056832021	90496156801
24	8570984	21025920334	473952342863	1265538103128	878627719204	213115694630
25	99608768	290672080092				
26	65098841	471406008869				
27	659784981	5907955340082				
28	480344748	10399034443419				
29	4374216671	120387859798419				
30	3475975951	226885800554545				

Table 29. $\pi^{(6)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
21	16664582	0	0	0	0
22	35604320	0	0	0	0
23	8516021310	261485631	0	0	0
24	19516357614	571565220	0	0	0

Table 30. $\pi^{(7)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
10	13	0	0	0	0	0
11	0	0	0	0	0	0
12	167	130	0	0	0	0
13	0	36	0	0	0	0
14	1566	4848	1261	0	0	0
15	54	2142	900	0	0	0
16	12652	126235	98875	12831	0	0
17	1436	81672	95646	15360	0	0
18	94214	2890749	5231126	1816315	141947	0
19	20570	2506758	6468612	2682468	232044	0
20	670432	62721700	238884419	168087047	32973639	1739842
21	221828	67979018	358882454	294489224	62574442	3420240
22	4656564	1328428920	10234310356	13113683286	4870122569	620213944
23	2050070	1709692876	17954515252	26385524868	10443625264	1357045652
24	31929088	27823812351	426328314404	943453947145	593319098550	137299794387
25	17292362	40985235874				
26	217411992	579807569634				
27	137743254	951779591064				
28	1474682620	12057289021685				
29	1056607484	21626765622908				
30	9980819692	250613874171918				

Table 31. $\pi^{(7)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
22	23862930	0	0	0	0
23	51687000	0	0	0	0
24	12416413342	367624101	0	0	0

Table 32. $\pi^{(8)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
11	15	0	0	0	0	0
12	0	0	0	0	0	0
13	222	165	0	0	0	0
14	0	42	0	0	0	0
15	2340	6833	1700	0	0	0
16	63	2831	1134	0	0	0
17	20691	193106	143691	17895	0	0
18	1922	118687	132247	20398	0	0
19	165469	4712394	8046381	2670249	200748	0
20	30951	3933154	9600339	3813193	318612	0
21	1245781	107500565	383663701	256970164	48366936	2458650
22	366583	113542157	562901848	440713133	90239128	4781868
23	9047308	2370454739	16995221503	20643195267	7336799080	898762182
24	3651309	3005915827	29419819728	41074183956	15624060003	1964400454
25	64301059	51321493165				
26	32704796	75185702070				
27	450992816	1099638379068				
28	273507135	1809278327735				
29	3136693513	23418087499989				
30	2183903932	42372199889391				

Table 33. $\pi^{(8)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
23	33360840	0	0	0	0
24	72691486	0	0	0	0

Table 34. $\pi^{(9)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
12	17	0	0	0	0	0
13	0	0	0	0	0	0
14	285	204	0	0	0	0
15	0	48	0	0	0	0
16	3346	9286	2227	0	0	0
17	72	3616	1392	0	0	0
18	32268	283378	201817	24276	0	0
19	2480	165466	176680	26336	0	0
20	276784	7353849	11934449	3808709	276930	0
21	44464	5894748	13707358	5245928	425392	0
22	2205594	176290082	593803816	380845303	69126106	3400408
23	575658	180731496	847545592	636897930	126252520	6510576
24	16770778	4048321273	27202526546	31505216907	10767477738	1273967484
25	6167028	5031404514				
26	123744344	90657805200				
27	58633798	131276628302				
28	895318616	1998767390386				
29	515053396	3274111050700				
30	6393143772	43624309476410				

Table 35. $\pi^{(9)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
24	45780711	0	0	0	0

Table 36. $\pi^{(10)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
13	19	0	0	0	0	0
14	0	0	0	0	0	0
15	356	247	0	0	0	0
16	0	54	0	0	0	0
17	4616	12255	2850	0	0	0
18	81	4497	1674	0	0	0
19	48359	401907	275617	32186	0	0
20	3110	223161	229665	33246	0	0
21	444151	11065086	17172832	5297255	374167	0
22	61541	8511804	18968217	7028469	555264	0
23	3745051	278495005	890870225	549912277	96668670	4619451
24	866765	276504515	1234168142	894541325	172337878	8675838
25	29835862	6659590982				
26	9963655	8084548819				
27	228812507	154303888066				
28	100468672	219927474798				
29	1709867120	3502340628584				
30	927055673	5684652308909				

Table 37. $\pi^{(11)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
14	21	0	0	0	0	0
15	0	0	0	0	0	0
16	435	294	0	0	0	0
17	0	60	0	0	0	0
18	6182	15788	3577	0	0	0
19	90	5474	1980	0	0	0
20	70068	553997	367647	41853	0	0
21	3812	292924	291922	41200	0	0
22	687658	16142157	24082872	7210887	496629	0
23	82614	11916874	25576056	9212068	711300	0
24	6130948	426057336	1301357573	776466251	132681925	6179866
25	1259920	409182534				
26	51185656	10605055038				
27	15500930	12547916024				
28	408295664	254252978969				
29	165576770	355652651962				
30	3154240824	5942906034046				

Table 38. $\pi^{(12)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
15	23	0	0	0	0	0
16	0	0	0	0	0	0
17	522	345	0	0	0	0
18	0	66	0	0	0	0
19	8076	19933	4416	0	0	0
20	99	6547	2310	0	0	0
21	98627	745400	480655	53521	0	0
22	4586	375907	364171	50270	0	0
23	1032001	22931034	33032951	9634685	649014	0
24	108115	16254798	33738607	11851433	896764	0
25	9718849	633875781				
26	1777443	588573593				
27	85015672	16413684549				
28	23343425	18902891979				
29	705628035	407117163519				
30	263773508	557778339126				

Table 39. $\pi^{(13)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
16	25	0	0	0	0	0
17	0	0	0	0	0	0
18	617	400	0	0	0	0
19	0	72	0	0	0	0
20	10330	24738	5375	0	0	0
21	108	7716	2664	0	0	0
22	135396	982316	617581	67450	0	0
23	5432	473262	447132	60528	0	0
24	1506996	31831449	44441691	12664449	836580	0
25	138476	21682704				
26	14972286	920256950				
27	2443958	826095332				
28	137163726	24771314463				
29	34172064	27745500038				
30	1184699784	635464161474				

Table 40. $\pi^{(14)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
17	27	0	0	0	0	0
18	0	0	0	0	0	0
19	720	459	0	0	0	0
20	0	78	0	0	0	0
21	12976	30251	6462	0	0	0
22	117	8981	3042	0	0	0
23	181863	1271393	781557	83916	0	0
24	6350	586141	541525	72046	0	0
25	2148091	43300734				
26	174129	28370008				
27	22484007	1307398301				
28	3286393	1134897871				
29	215579698	36553318000				
30	48795643	39803399843				

Table 41. $\pi^{(15)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
18	29	0	0	0	0	0
19	0	0	0	0	0	0
20	831	522	0	0	0	0
21	0	84	0	0	0	0
22	16046	36520	7685	0	0	0
23	126	10342	3444	0	0	0
24	239644	1619727	975907	103211	0	0
25	7340	715696				
26	2996878	57857661				
27	215506	36498414				
28	32999272	1821902828				
29	4333980	1529986690				
30	330883356	52861529660				

Table 42. $\pi^{(16)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
19	31	0	0	0	0	0
20	0	0	0	0	0	0
21	950	589	0	0	0	0
22	0	90	0	0	0	0
23	19572	43593	9052	0	0	0
24	135	11799	3870	0	0	0
25	310483	2034862				
26	8402	863079				
27	4101605	76086282				
28	263039	46261914				
29	47441197	2495325493				
30	5618255	2028345509				

Table 43. $\pi^{(17)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
20	33	0	0	0	0	0
21	0	0	0	0	0	0
22	1077	660	0	0	0	0
23	0	96	0	0	0	0
24	23586	51518	10571	0	0	0
25	144	13352				
26	396252	2524790				
27	9536	1029442				
28	5517688	98639769				
29	317160	57866788				
30	66938146	3364751402				

Table 44. $\pi^{(18)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
21	35	0	0	0	0	0
22	0	0	0	0	0	0
23	1212	735	0	0	0	0
24	0	102	0	0	0	0
25	28120	60343				
26	153	15001				
27	498951	3097951				
28	10742	1215937				
29	7308223	126244254				
30	378301	71531604				

Table 45. $\pi^{(19)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
22	37	0	0	0	0	0
23	0	0	0	0	0	0
24	1355	814	0	0	0	0
25	0	108				
26	33206	70116				
27	162	16746				
28	620708	3763233				
29	12020	1423716				
30	9544498	159702669				

Table 46. $\pi^{(20)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
23	39	0	0	0	0	0
24	0	0	0	0	0	0
25	1506	897				
26	0	114				
27	38876	80885				
28	171	18587				
29	763779	4529972				
30	13370	1653931				

Table 47. $\pi^{(21)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
24	41	0	0	0	0	0
25	0	0				
26	1665	984				
27	0	120				
28	45162	92698				
29	180	20524				
30	930548	5407952				

Table 48. $\pi^{(22)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
25	43	0				
26	0	0				
27	1832	1075				
28	0	126				
29	52096	105603				
30	189	22557				

Table 49. $\pi^{(23)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
26	45	0				
27	0	0				
28	2007	1170				
29	0	132				
30	59710	119648				

Table 50. $\pi^{(24)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
27	47	0				
28	0	0				
29	2190	1269				
30	0	138				

Table 51. $\pi^{(25)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
28	49	0				
29	0	0				
30	2381	1372				

Table 52. $\pi^{(26)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
29	51	0				
30	0	0				

Table 53. $\pi^{(27)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
30	53	0				

Table 54. $r^{(2)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
5	3	0	0	0	0	0
6	0	0	0	0	0	0
7	12	15	0	0	0	0
8	0	12	0	0	0	0
9	60	267	106	0	0	0
10	24	288	180	0	0	0
11	303	4563	5361	966	0	0
12	240	6396	10364	2300	0	0
13	1719	80562	217776	104991	10755	0
14	1716	134712	478644	270600	30048	0
15	10563	1470573	8409505	8224477	2116374	141271
16	11592	2797044	20637648	23086020	6314616	421740
17	66234	27525930	322435857	577027293	281180154	45138237
18	80076	57973356	867563808	1750871564	896066364	143247884
19	418683	526143030	12452324349	38804133468	32073010698	9328659543
20	555000	1202297328	36173113884	126006015372	108471468768	31280759520
21	2684142	10231531455	486614446239	2571275766429	3407644412256	1604854311453
22	3816852	24979291128	1507407294648	8852990471220	12141096419148	5645080129680
23	17421909	201733306620	19257248028189	170009849583132	349710516131892	251422365988539
24	26177568	520253143860	62995849579032	615792917891140	1303856142773740	921549979727688
25	114000531	4022258380917				
26	179542332	10863630641604				
27	750753132	80939701867716				
28	1231146888	227418680578320				
29	4971764847	1641332887015689				
30	8440422192	4772048378651280				

Table 55. $r^{(2)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
17	2136990	0	0	0	0
18	6468936	0	0	0	0
19	1030433256	36572274	0	0	0
20	3300957744	109065744	0	0	0
21	311604068736	25259978988	698531550	0	0
22	1046518704060	79219722012	2022777504	0	0
23	76615915096944	10683829264500	664458163260	14729974326	0
24	267648600729136	34881559715964	2008373490512	41188385208	0

Table 56. $r^{(3)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
6	2	0	0	0	0	0
7	0	0	0	0	0	0
8	16	16	0	0	0	0
9	0	20	0	0	0	0
10	102	394	130	0	0	0
11	50	682	372	0	0	0
12	576	7880	8028	1208	0	0
13	628	18004	26474	5280	0	0
14	3416	154722	370110	152802	13034	0
15	5262	420622	1374836	707716	72188	0
16	22184	3083620	15691856	13285076	2908980	162592
17	38584	9352414	63822546	65459156	16508102	1024104
18	148988	62293500	648199468	1006875428	421637034	57748488
19	277306	203837192	2826310104	5243925520	2480313612	368793664
20	1004632	1272386720	26631267660	71909411000	51257435976	12838224344
21	1984214	4401961634	122611887446	392663172244	312669832458	83958270270
22	6808300	26224063274	1097096101720	5002557027938	5716995562854	2328250510140
23	14077538	94628425518	5274717758732	28431403676998	36054110570432	15616820553396
24	46426496	544368343280	45454125260992	344425087809732	609554061849608	379398466937604
25	99256696	2029717393804				
26	317818172	11365001111056				
27	697456440	43494868547454				
28	2181058760	238375197808428				
29	4889200298	931819468205876				
30	14997374392	5018983220348986				

Table 57. $r^{(3)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
18	2323728	0	0	0	0
19	15553776	0	0	0	0
20	1223086496	37660320	0	0	0
21	8287529164	256640096	0	0	0
22	393015780140	27934853862	685064116	0	0
23	2711482261802	192827785048	4631723652	0	0
24	100936816144340	12420541690612	690207948060	13850676448	0

Table 58. $r^{(4)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
7	7	0	0	0	0	0
8	0	0	0	0	0	0
9	54	51	0	0	0	0
10	0	36	0	0	0	0
11	356	1347	418	0	0	0
12	76	1438	708	0	0	0
13	2211	27996	26549	3941	0	0
14	1232	42046	56922	10440	0	0
15	13693	557042	1235995	494669	42912	0
16	12136	1061546	3195032	1517382	145468	0
17	88085	11116475	52434783	42716032	9335298	537780
18	99480	24998120	156271770	147841802	35001696	2072848
19	583192	224435463	2159630873	3214139957	1331331688	185085700
20	764384	568813432	7182627060	12267303394	5438753990	769974992
21	3913879	4582977579	88378362030	227826707783	159817381368	40314134511
22	5719500	12707488472	320435224358	941692258378	701557424876	179025329652
23	26484064	94486203288	3626378214904	15730660101511	17631452388878	7199636720001
24	42072480	280881418960	14090890782836	69436772680542	82205291305560	33784062841664
25	180404733	1963301674531				
26	305839184	6168021422304				
27	1234935531	41054440588232				
28	2205438520	134883423136018				
29	8483759792	862936444427272				
30	15811361528	2941631612525142				

Table 59. $r^{(4)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
19	7703472	0	0	0	0
20	31319416	0	0	0	0
21	3931003835	125038266	0	0	0
22	16999897130	511311616	0	0	0
23	1236798035204	90221301603	2277961589	0	0
24	5633562148364	388379231186	9109343892	0	0

Table 60. $r^{(5)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
8	4	0	0	0	0	0
9	0	0	0	0	0	0
10	58	46	0	0	0	0
11	0	40	0	0	0	0
12	512	1672	448	0	0	0
13	100	2090	936	0	0	0
14	3768	41444	34488	4486	0	0
15	2040	71872	89134	15160	0	0
16	25808	921606	1806880	641346	48924	0
17	23564	2026170	5574602	2457996	221688	0
18	176096	19896440	83121094	60358756	11729024	594288
19	215826	51563496	293454976	257466024	57326914	3220080
20	1218516	426979382	3635795858	4828612720	1787755678	221049594
21	1780232	1241946768	14214493996	22464328540	9357299200	1255883130
22	8511948	9165645106	155970915360	358314700736	225105451414	50800964790
23	13967518	28983926094	659480292168	1788110042782	1249730137122	302138640808
24	59736408	197088785006	6649843869230	25634622474954	25728820744990	9426231551894
25	106509028	663410002162				
26	420323640	4245858974942				
27	797307884	14993787484376				
28	2961921852	91619917908366				
29	5892890654	335965300929438				
30	20887439776	1979819691816772				

Table 61. $r^{(5)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
20	8111908	0	0	0	0
21	48638424	0	0	0	0
22	4414525188	124647978	0	0	0
23	27320123866	783692712	0	0	0
24	1452030154534	94932169306	2150437088	0	0

Table 62. $r^{(6)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
9	11	0	0	0	0	0
10	0	0	0	0	0	0
11	132	105	0	0	0	0
12	0	60	0	0	0	0
13	1184	3771	1000	0	0	0
14	128	3424	1444	0	0	0
15	9199	96119	77325	10094	0	0
16	3124	127474	149292	24144	0	0
17	66355	2193208	4099834	1435207	111421	0
18	41448	3850004	9947986	4161788	360964	0
19	466403	48214509	190303297	134960205	26296982	1368005
20	421064	103593520	549459414	455744362	97297212	5301400
21	3272862	1048188206	8373211669	10789166283	3971592938	497983069
22	3741048	2607720328	27604249712	41073572616	16356525060	2122781616
23	23059691	22733534220	360630405632	799611796857	496708177415	112861166915
24	30950068	63045503272	1317337306400	3348463774668	2231021986444	520240499072
25	163038648	493217132850				
26	245634696	1485333095472				
27	1155273873	10711876775928				
28	1898422136	34390850906318				
29	8196381995	232906931549267				
30	14407634436	786626340102656				

Table 63. $r^{(6)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
21	18843084	0	0	0	0
22	80212944	0	0	0	0
23	10015392896	291936913	0	0	0
24	45766091548	1285849228	0	0	0

Table 64. $r^{(7)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
10	6	0	0	0	0	0
11	0	0	0	0	0	0
12	128	90	0	0	0	0
13	0	60	0	0	0	0
14	1522	4306	1030	0	0	0
15	150	4266	1692	0	0	0
16	14012	130042	94872	11272	0	0
17	4332	182964	201514	30960	0	0
18	113240	3317564	5625980	1802320	127554	0
19	65930	6121534	14796474	5871274	487332	0
20	859496	78921330	282092266	183371884	32754638	1549518
21	743908	177869080	876684166	687555954	140263172	7344672
22	6360436	1820553832	13132541616	15502861618	5243777518	601546460
23	7149314	4749641454	46410862022	65058238496	24705608676	3077728590
24	46637248	41377965724	590646115468	1197601630526	683884550288	142607096940
25	62694238	120287159050				
26	340649368	932863190526				
27	520129314	2942051744274				
28	2482125884	20926087842756				
29	4163391220	70256436220362				
30	18047370304	467797579924514				

Table 65. $r^{(7)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
22	20636548	0	0	0	0
23	111950328	0	0	0	0
24	11549079642	304966884	0	0	0

Table 66. $r^{(8)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
11	15	0	0	0	0	0
12	0	0	0	0	0	0
13	246	177	0	0	0	0
14	0	84	0	0	0	0
15	2876	8043	1936	0	0	0
16	180	6242	2388	0	0	0
17	27519	246212	176745	21175	0	0
18	5912	284198	300966	44780	0	0
19	233385	6445982	10596499	3371581	242056	0
20	100928	10089506	23288578	8912670	720424	0
21	1845757	157158801	538477043	344296686	61650610	2974748
22	1251128	308985908	1442394246	1086055304	214987600	11014584
23	14073836	3701468305	25365861171	29246670775	9841496998	1140183636
24	12954740	8626044740	79177357630	106059203796	38949000104	4731967116
25	105417439	85615405119				
26	120393180	226724970456				
27	782590956	1959710902768				
28	1045226888	5720733356218				
29	5779657189	44557462523871				
30	8675875332	140262928497856				

Table 67. $r^{(8)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
23	40054722	0	0	0	0
24	168966820	0	0	0	0

Table 68. $r^{(9)}_{m,\delta}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
12	8	0	0	0	0	0
13	0	0	0	0	0	0
14	226	148	0	0	0	0
15	0	80	0	0	0	0
16	3436	8752	1952	0	0	0
17	200	7210	2640	0	0	0
18	38628	313666	209148	23380	0	0
19	7504	372416	376814	54000	0	0
20	366256	9133810	13924418	4145316	278000	0
21	143848	14516626	31845086	11713378	915824	0
22	3141068	240496698	761591474	455622222	76428546	3435550
23	1961590	478108124	2107133662	1519778710	290489100	14429536
24	25421244	6004389082	37875144900	40803261670	12871734380	1393553952
25	21929740	14143818726				
26	199120480	145413163448				
27	216632478	389621851240				
28	1530667260	3455385009914				
29	1974632708	10218346781498				
30	11630685456	81062344414750				

Table 69. $r^{(9)}_{m,\delta}$

m	$\delta=8$	$\delta=9$	$\delta=10$	$\delta=11$	$\delta=12$
24	45397140	0	0	0	0

Table 70. $r_{m,\delta}^{(10)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
13	19	0	0	0	0	0
14	0	0	0	0	0	0
15	396	267	0	0	0	0
16	0	108	0	0	0	0
17	5768	14667	3310	0	0	0
18	232	9892	3540	0	0	0
19	66531	526587	348569	39326	0	0
20	9596	534106	525624	73716	0	0
21	658447	15681988	23408614	6944671	470991	0
22	202144	21890044	46395636	16627272	1275736	0
23	5894515	423798251	1299642891	767994261	129033850	5883901
24	2995896	756335696	3195791818	2235604108	417833868	20419732
25	49489794	10842464020				
26	35935328	23354245068				
27	399244291	268351067698				
28	376294192	667804349640				
29	3141942984	6500423848630				
30	3598237012	18087878452598				

Table 71. $r_{m,\delta}^{(11)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
14	10	0	0	0	0	0
15	0	0	0	0	0	0
16	352	220	0	0	0	0
17	0	100	0	0	0	0
18	6558	15466	3290	0	0	0
19	250	10922	3780	0	0	0
20	87936	642836	401796	43000	0	0
21	11556	661348	628234	85600	0	0
22	967256	21149928	29755618	8387574	540306	0
23	268790	29524014	60122642	20891832	1561740	0
24	9374028	615006672	1771150306	993883076	158851244	6865746
25	4338696	1091706032				
26	83667868	16647772640				
27	55926182	35630285146				
28	707929888	430978334272				
29	621880718	1066784975226				
30	5785321844	10831341361474				

Table 72. $r_{m,\delta}^{(12)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
15	23	0	0	0	0	0
16	0	0	0	0	0	0
17	582	375	0	0	0	0
18	0	132	0	0	0	0
19	10196	24147	5206	0	0	0
20	284	14374	4900	0	0	0
21	138731	996532	621261	67081	0	0
22	14176	899086	836946	112320	0	0
23	1583781	33394122	46294451	13027421	846608	0
24	356664	41873962	83056140	28306838	2086788	0
25	15998341	995892731				
26	6193992	1616606484				
27	148429584	27657829123				
28	85192508	54932969900				
29	1298770487	733287826193				
30	1002009628	1705621594848				

Table 73. $r_{m,\delta}^{(13)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
16	12	0	0	0	0	0
17	0	0	0	0	0	0
18	506	306	0	0	0	0
19	0	120	0	0	0	0
20	11192	24904	5120	0	0	0
21	300	15402	5112	0	0	0
22	175264	1178600	701808	72698	0	0
23	16488	1070880	968974	127080	0	0
24	2208848	43346846	57306860	15489502	967260	0
25	452228	53960074				
26	24064072	1385097316				
27	8476890	2218798644				
28	237072324	40597920754				
29	124612560	79446977894				
30	2177283104	1124266627640				

Table 74. $r_{m,\delta}^{(14)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
17	27	0	0	0	0	0
18	0	0	0	0	0	0
19	804	501	0	0	0	0
20	0	156	0	0	0	0
21	16496	36987	7708	0	0	0
22	336	19688	6468	0	0	0
23	259751	1726311	1027989	107366	0	0
24	19652	1401026	1248612	161960	0	0
25	3379115	64418512				
26	576056	73175604				
27	38269747	2107668193				
28	11516392	3127726016				
29	391793158	63373541574				
30	179627912	116278758388				

Table 75. $r_{m,\delta}^{(15)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
18	14	0	0	0	0	0
19	0	0	0	0	0	0
20	688	406	0	0	0	0
21	0	140	0	0	0	0
22	17642	37522	7518	0	0	0
23	350	20650	6636	0	0	0
24	316948	1992534	1142688	115416	0	0
25	22300	1622132				
26	4522264	81119956				
27	705634	91144990				
28	54997600	2831949190				
29	15109676	4131333512				
30	596459876	89621999512				

Table 76. $r_{m,\delta}^{(16)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
19	31	0	0	0	0	0
20	0	0	0	0	0	0
21	1062	645	0	0	0	0
22	0	180	0	0	0	0
23	25004	53691	10900	0	0	0
24	388	25834	8244	0	0	0
25	448359	2797164				
26	26024	2061814				
27	6570593	115151478				
28	871888	119441314				
29	82820669	4108450397				
30	19766424	5605767956				

Table 77. $r_{m,\delta}^{(17)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
20	16	0	0	0	0	0
21	0	0	0	0	0	0
22	898	520	0	0	0	0
23	0	160	0	0	0	0
24	26212	53776	10560	0	0	0
25	400	26666				
26	532332	3166742				
27	28992	2336224				
28	8505776	141492234				
29	1040480	144902754				
30	114631028	5365597606				

Table 78. $r_{m,\delta}^{(18)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
21	35	0	0	0	0	0
22	0	0	0	0	0	0
23	1356	807	0	0	0	0
24	0	204	0	0	0	0
25	36056	74763				
26	440	32812				
27	726459	4301307				
28	33292	2903338				
29	11869703	193766492				
30	1255728	184829436				

Table 79. $r_{m,\delta}^{(19)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
22	18	0	0	0	0	0
23	0	0	0	0	0	0
24	1136	648	0	0	0	0
25	0	180				
26	37206	74122				
27	450	33450				
28	843768	4793856				
29	36564	3234276				
30	14953112	233327360				

Table 80. $r_{m,\delta}^{(20)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
23	39	0	0	0	0	0
24	0	0	0	0	0	0
25	1686	987				
26	0	228				
27	49988	100707				
28	492	40622				
29	1119091	6341932				
30	41456	3947486				

Table 81. $r_{m,\delta}^{(21)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
24	20	0	0	0	0	0
25	0	0				
26	1402	790				
27	0	200				
28	50928	99016				
29	500	41002				
30	1276616	6977036				

Table 82. $r_{m,\delta}^{(22)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
25	43	0				
26	0	0				
27	2052	1185				
28	0	252				
29	67136	132027				
30	544	49264				

Table 83. $r_{m,\delta}^{(23)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
26	22	0				
27	0	0				
28	1696	946				
29	0	220				
30	67682	128914				

Table 84. $r_{m,\delta}^{(24)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
27	47	0				
28	0	0				
29	2454	1401				
30	0	276				

Table 85. $r_{m,\delta}^{(25)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
28	24	0				
29	0	0				
30	2018	1116				

Table 86. $r_{m,\delta}^{(26)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
29	51	0				
30	0	0				

Table 87. $r_{m,\delta}^{(27)}$

m	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$	$\delta=6$	$\delta=7$
30	26	0				

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